

On \mathcal{L} -Injective Modules

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Abstract

Let $\mathcal{M} = \{(M, N, f, Q) \mid M, N, Q \in R\text{-Mod}, N \leq M, f \in \text{Hom}_R(N, Q)\}$ and let \mathcal{L} be a nonempty subclass of \mathcal{M} . Jirásko introduced the concepts of \mathcal{L} -injective module as a generalization of injective module as follows: a module Q is said to be \mathcal{L} -injective if for each $(B, A, f, Q) \in \mathcal{L}$, there exists a homomorphism $g : B \rightarrow Q$ such that $g(a) = f(a)$, for all $a \in A$. The aim of this paper is to study \mathcal{L} -injective modules and some related concepts.

Key words and phrases: Injective module; Baer's criterion; Generalized Fuchs criterion; τ -Injective module; P-filter; Hereditary torsion theory; τ -dense; Preradicals; M-injective module; Quasi-injective; Natural class.

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1 Introduction

Injective modules can be traced to Baer [4] who studied divisible Abelian groups. Eckmann and Schopf [12] introduced the terminology "injective". An R -module M is said to be injective if, for any module B , every homomorphism $f : A \rightarrow M$, where A is any submodule of B , extends to a homomorphism $g : B \rightarrow M$. Many authors interested in this class of modules, for example, Matlis [19], Faith [13] and Page and Zhou [21]. Also, many authors tried to generalize the concept of injective module, for example, Johnson and Wong introduced quasi-injective module in [17]. The notion of M -injective module was introduced in [3]. The notion of τ -injective module was studied in [23] and the notion of soc-injective module was introduced in [1]. Also, Zeyada in [25] introduced the concept of s -injective module.

Let M and N be modules. Recall that N is said to be M -injective if every homomorphism from a submodule of M to N extends to a homomorphism from M to N [3]. A module M is said to be quasi-injective if M is M -injective.

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. A submodule B of a module A is said to be τ -dense in A if A/B is τ -torsion (i.e. $A/B \in \mathcal{T}$). A submodule A of a module B is said to be τ -essential in B if it is τ -dense and essential in B . A torsion theory τ is said to be noetherian if for every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R with $I_\infty = \bigcup_{j=1}^\infty I_j$ a τ -dense left ideal in R , there exists a positive integer n such that I_n is τ -dense in R . A module M is said to be τ -injective if every homomorphism from a τ -dense submodule of B to M extends to a homomorphism from B to M , where B is any module. Let M be an R -module. A τ -injective envelope (or τ -injective hull) of M is a τ -injective module E which is a τ -essential extension of M [7]. Every R -module M has a τ -injective envelope and it is unique up to isomorphism [9]. We use the notation $E_\tau(M)$ to stand for an τ -injective envelope of M . A τ -injective module E is said to be Σ - τ -injective if $E^{(A)}$ is τ -injective for any index set A ; E is said to be countably Σ - τ -injective in case $E^{(C)}$ is τ -injective for any countable index set C . Let E and M be modules. Then E is said to be τ - M -injective if any homomorphism from a τ -dense submodule of M to E extends to a homomorphism from M to E . A module E is said to be τ -quasi-injective if E is τ - E -injective.

Let $\mathcal{M} = \{(M, N, f, Q) \mid M, N, Q \in R\text{-Mod}, N \leq M, f \in \text{Hom}_R(N, Q)\}$ and let \mathcal{L} be a nonempty subclass of \mathcal{M} . Jirásko in [16] introduced the concepts of \mathcal{L} -injective module as a generalization of injective module as follows: a module Q is said to be \mathcal{L} -injective if for each $(B, A, f, Q) \in \mathcal{L}$, there exists a homomorphism $g : B \rightarrow Q$ such that $(g \upharpoonright A) = f$. Clearly, injective module and all its generalizations are special cases of \mathcal{L} -injectivity. The aim of this article is to study \mathcal{L} -injectivity and some related concepts.

In section two, we give some properties and characterizations of \mathcal{L} -injective modules. For example, in Theorem 2.12 we give a version of Baer's criterion for \mathcal{L} -injectivity. Also, in Theorem 2.15 we extend a characterization due to [24, Theorem 2, p. 8] of \mathcal{L} -injective modules over commutative Noetherian rings.

In section three we introduce the concepts of \mathcal{L} - M -injective module and s - \mathcal{L} - M -injective module as generalizations of M -injective modules and give some results about them. For examples, in Theorem 3.5 we prove that if \mathcal{L} is a nonempty subclass of \mathcal{M} satisfy conditions (α) , (β) and (γ) and $M, Q \in R\text{-Mod}$ such that M satisfies condition $(E_{\mathcal{L}})$, then Q is \mathcal{L} - M -injective if and only if $f(M) \leq Q$, for all $f \in \text{Hom}_R(E_{\mathcal{L}}(M), E_{\mathcal{L}}(Q))$ with $(M, L, f \upharpoonright L, Q) \in \mathcal{L}$ where $L = \{m \in M \mid f(m) \in Q\} = M \cap f^{-1}(Q)$. Also, in Proposition 3.13 we generalize [7, Proposition 14.12, p. 66], [6, Proposition 1, p. 1954] and Fuchs's result in [14]. Moreover, our version of the Generalized Fuchs Criterion is given in Proposition 3.14 in which we prove that if \mathcal{L} is a nonempty subclass of \mathcal{M} satisfy conditions (α) and (μ) and $M, Q \in R\text{-Mod}$ such that M satisfies condition (\mathcal{L}) . Then a module Q is s - \mathcal{L} - M -injective if and only if for each $(R, I, f, Q) \in \mathcal{L}$ with $\ker(f) \in \Omega(M)$, there exists an element $x \in Q$ such that $f(a) = ax, \forall a \in I$.

In section four we study direct sums of \mathcal{L} -injective modules. In this section we prove some results, for example, in Proposition 4.3 we prove that for any family $\{E_{\alpha}\}_{\alpha \in A}$ of \mathcal{L} -injective modules, where A is an infinite index set, if \mathcal{L} satisfies conditions (α) , (μ) and (δ) and $\bigoplus_{\alpha \in C} E_{\alpha}$ is an \mathcal{L} -injective module for any countable subset C of A , then $\bigoplus_{\alpha \in A} E_{\alpha}$ is an \mathcal{L} -injective module. In Theorem 4.10 we prove that for any nonempty subclass \mathcal{L} of \mathcal{M} which satisfies conditions (α) and (δ) and for any nonempty class \mathcal{K} of modules closed under isomorphic copies and \mathcal{L} -injective hulls, if the direct sum of any family $\{E_i\}_{i \in \mathbb{N}}$ of \mathcal{L} -injective R -modules in \mathcal{K} is \mathcal{L} -injective, then every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R in $H_{\mathcal{K}}(R)$ with $I_{\infty} = \bigcup_{j=1}^{\infty} I_j$ s - \mathcal{L} -dense in R , terminates. Also, in Theorem 4.12 we generalize results in [21, p. 643] and [9, Proposition 5.3.5, p. 165] in which we prove that for any nonempty subclass \mathcal{L} of \mathcal{M} which satisfies conditions (α) , (μ) , (δ) and (I) and for any nonempty class \mathcal{K} of modules closed under isomorphic copies and submodules, if every ascending chain $J_1 \subseteq J_2 \subseteq \dots$ of left ideals of R such that $(J_{i+1}/J_i) \in \mathcal{K}$, $\forall i \in \mathbb{N}$ and $J_{\infty} = \bigcup_{i=1}^{\infty} J_i$ s - \mathcal{L} -dense in R terminates, then every direct sum of \mathcal{L} -injective modules in \mathcal{K} is \mathcal{L} -injective.

Finally, in section five we introduce the concept of Σ - \mathcal{L} -injectivity as a generalization of Σ -injectivity and Σ - τ -injectivity and prove Theorem 5.4 in which we generalize Faith's result [13, Proposition 3, p. 184] and [7, Theorem 16.16, p. 98].

Throughout this article, unless otherwise specified, R will denote an associative ring with non-zero identity, and all modules are left unital R -modules. By a class of modules we mean a non-empty class of modules. The class of all left R -modules is denoted by $R\text{-Mod}$ and by \mathfrak{R} we mean the set $\{(M, N) \mid N \leq M, M \in R\text{-Mod}\}$, where $N \leq M$ is a notation means N is a submodule of M . Given a family of modules $\{M_i\}_{i \in I}$, for each $j \in I$, $\pi_j : \bigoplus_{i \in I} M_i \rightarrow M_j$ denotes the canonical projection homomorphism. Let M be a module and let Y be a subset of M . The left annihilator of Y in R will be denoted by $l_R(Y)$, i.e., $l_R(Y) = \{r \in R \mid ry = 0, \forall y \in Y\}$. Given $a \in M$, let $(Y : a)$ denote the set $\{r \in R \mid ra \in Y\}$, and let $\text{ann}_R(a) := (0 : a)$. The right annihilator of a subset I of R in M will be denoted by $r_M(I)$, i.e., $r_M(I) = \{m \in M \mid rm = 0, \forall r \in I\}$. The class $\{I \mid I \text{ is a left ideal of } R \text{ such that } \text{ann}_R(m) \subseteq I, \text{ for some } m \in M\}$ will be denoted by $\Omega(M)$. Finally, the injective envelope of a module M will be denoted by $E(M)$.

2 Some Properties and Characterizations of \mathcal{L} -Injective Modules

Let $\mathcal{M} = \{(M, N, f, Q) \mid M, N, Q \in R\text{-Mod}, N \leq M, f \in \text{Hom}_R(N, Q)\}$ and let \mathcal{L} be a nonempty subclass of \mathcal{M} . Jirásko in [16] introduced the concept of \mathcal{L} -injective module as a generalization

of injective module as follows: a module Q is said to be \mathcal{L} -injective if every diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ f \downarrow & & \\ Q & & \end{array}$$

with $(M, N, f, Q) \in \mathcal{L}$, can be completed to a commutative diagram.

For every $(B, A, f, Q) \in \mathcal{M}$, Jirásko in [16] defined $r_{\mathcal{L}}(B, A, f, Q)$ and $s_{\mathcal{L}}(B, A, f, Q)$ as follows: $r_{\mathcal{L}}(B, A, f, Q)$ ($s_{\mathcal{L}}(B, A, f, Q)$) is the submodule of B generated by all the $g(M)$, $g \in \text{Hom}_R(M, B)$ for which there exists a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ h \downarrow & & \downarrow g \\ A & \xrightarrow{i} & B \\ f \downarrow & & \\ Q & & \end{array}$$

with $(M, N, fh, Q) \in \mathcal{L}$ (and $N = g^{-1}(A)$).

We will use the following abbreviations: $r_{\mathcal{L}}(B, Q, I_Q, Q) = r_{\mathcal{L}}(B, Q)$, $s_{\mathcal{L}}(B, Q, I_Q, Q) = s_{\mathcal{L}}(B, Q)$, $r_{\mathcal{L}}(E(Q), Q) = r_{\mathcal{L}}(Q)$ and $s_{\mathcal{L}}(E(Q), Q) = s_{\mathcal{L}}(Q)$.

A submodule N of a module M is said to be \mathcal{L} -dense in M if $M \subseteq N + r_{\mathcal{L}}(E(M), N)$ [16].

Jirásko in [16] introduced the concepts of \mathcal{L} -injective envelope (or \mathcal{L} -injective hull) of a module M as follows: an \mathcal{L} -injective module E is said to be an \mathcal{L} -injective envelope of a module M if there is no proper \mathcal{L} -injective submodule of E containing M . If a module M has an \mathcal{L} -injective envelope and it is unique up to isomorphic then we will use the notation $E_{\mathcal{L}}(M)$ to stand for an \mathcal{L} -injective envelope of M .

In the class \mathcal{M} we will define the partial order \preceq in the following way:

$$(M, N, f, Q) \preceq (M', N', f', Q') \iff M = M', N \subseteq N', Q = Q', f' \upharpoonright N = f.$$

The following conditions on \mathcal{L} will be useful later, where \mathcal{L} always denotes a nonempty subclass of \mathcal{M} .

(α) $(M, N, f, Q) \in \mathcal{L}$, $(M, N', f', Q) \in \mathcal{M}$ and $(M, N, f, Q) \preceq (M, N', f', Q)$ implies $(M, N', f', Q) \in \mathcal{L}$,

(β) $(M, N, f, A) \in \mathcal{L}$, $i : A \rightarrow B$ implies $(M, N, if, B) \in \mathcal{L}$, where i is an inclusion homomorphism,

(γ) $(M, N, f, A) \in \mathcal{L}$, $g : A \rightarrow B$ an isomorphism, implies $(M, N, gf, B) \in \mathcal{L}$,

(δ) $(M, N, f, A) \in \mathcal{L}$, $g : A \rightarrow B$ a homomorphism, implies $(M, N, gf, B) \in \mathcal{L}$,

(λ) $(M, N, f, A) \in \mathcal{L}$, $g : A \rightarrow B$ a split epimorphism, implies $(M, N, gf, B) \in \mathcal{L}$,

(μ) $(M, N, f, Q) \in \mathcal{L}$, implies $(R, (N : x), f_x, Q) \in \mathcal{L}$, $\forall x \in M$, where $f_x : (N : x) \rightarrow Q$ is a homomorphism define by $f_x(r) = f(rx)$, $\forall r \in (N : x)$.

Remark 2.1. ([16, p. 625]) If \mathcal{L} satisfies (γ), then the class of \mathcal{L} -injective modules is closed under isomorphism.

Theorem 2.2. ([16, Theorem 1.12, p. 625]) If the subclass \mathcal{L} of \mathcal{M} satisfies (α), (β) and (γ), then every module Q has an \mathcal{L} -injective envelope which is unique up to Q -isomorphism.

Proposition 2.3. If \mathcal{L} satisfies condition (β), then every direct summand of an \mathcal{L} -injective module is also \mathcal{L} -injective.

Proof. An easy check. □

Corollary 2.4. Let \mathcal{L} satisfy conditions (β) and (γ) and let $\{M_i\}_{i \in I}$ be any family of modules. If $\bigoplus_{i \in I} M_i$ is \mathcal{L} -injective, then M_i is \mathcal{L} -injective, $\forall i \in I$.

Proof. Since M_i is isomorphic to a direct summand of $\bigoplus_{i \in I} M_i$, this immediate by Proposition 2.3 and Remark 2.1. \square

The converse of Corollary 2.4 is not true in general, consider the following example:

Example 2.5. Let $\{T_i\}_{i \in I}$ be a family of rings with unit and let $R = \prod_{i \in I} T_i$ be the ring product of the family $\{T_i\}_{i \in I}$, where addition and multiplication are define componentwise. Let $A = \sqcup_{i \in I} T_i$ the direct sum of T_i , $\forall i \in I$. If each ${}_i T_i$ is injective, $\forall i \in I$ and I is infinite, then ${}_R A$ is a direct sum of injective modules, but ${}_R A$ is not itself injective, by [18, p. 140]. Hence we have that ${}_R A$ is a direct sum of \mathcal{L} -injective modules, but ${}_R A$ is not itself \mathcal{L} -injective where $\mathcal{L} = \mathcal{M}$.

Proposition 2.6. Let $\{M_i\}_{i \in I}$ be any family of modules. Then:

- (1) if $\prod_{i \in I} M_i$ is \mathcal{L} -injective and \mathcal{L} satisfies conditions (β) and (γ) , then M_i is \mathcal{L} -injective, $\forall i \in I$.
- (2) if M_i is \mathcal{L} -injective, $\forall i \in I$ and \mathcal{L} satisfies condition (λ) , then $\prod_{i \in I} M_i$ is \mathcal{L} -injective.

Proof. This is obvious. \square

Since $\prod_{i \in I} M_i = \bigoplus_{i \in I} M_i$ for any family $\{M_i\}_{i \in I}$ of modules with finite index set I , thus the following corollary is immediately from Proposition 2.6.

Corollary 2.7. Let \mathcal{L} satisfy condition (λ) and let $M_i\}_{i \in I}$ be any family of \mathcal{L} -injective modules. If I is a finite set, then $\bigoplus_{i \in I} M_i$ is \mathcal{L} -injective.

Remarks 2.8.

- (i) Let $\mathcal{L}_2 \subseteq \mathcal{L}_1$ be a nonempty subclasses of \mathcal{M} and let Q be any module. If Q is \mathcal{L}_1 -injective, then Q is \mathcal{L}_2 -injective.
- (ii) Let \mathcal{L} be a nonempty subclass of \mathcal{M} and let $\mathcal{L}_1 = \{(M, N, f, Q) \in \mathcal{L} \mid N \text{ is a direct summand of } M\}$. Then every module is \mathcal{L}_1 -injective.
- (iii) The converse of (i) is not true in general, for example let $\mathcal{L}_1 = \mathcal{M}$ and $\mathcal{L}_2 = \{(M, N, f, Q) \in \mathcal{L} \mid N \text{ is a direct summand of } M\}$, then $\mathcal{L}_2 \subseteq \mathcal{L}_1$. By (ii) every R -module is \mathcal{L}_2 -injective but, if R is not semisimple artinian, then not every module is \mathcal{L}_1 -injective (clearly \mathcal{L}_1 -injectivity = injectivity).

Now we will introduce the concept of P -filter as follows.

Definition 2.9. Let $\mathfrak{R} = \{(M, N) \mid N \leq M, M \in R\text{-Mod}\}$ and let ρ be a nonempty subclass of \mathfrak{R} . We say that ρ is a P -filter if ρ satisfies the following conditions:

- (i) if $(M, N) \in \rho$ and $N \leq K \leq M$, then $(M, K) \in \rho$;
- (ii) for all $M \in R\text{-Mod}$, $(M, M) \in \rho$;
- (iii) if $(M, N) \in \rho$, then $(R, (N : x)) \in \rho$, $\forall x \in M$.

Example 2.10. All of the following subclasses of \mathfrak{R} are P -filters.

- (1) $\rho_{\mathcal{T}} = \{(M, N) \in \mathfrak{R} \mid N \leq M \text{ such that } M/N \in \mathcal{T}, M \in R\text{-Mod}\}$, where \mathcal{T} is a nonempty class of modules closed under submodules and homomorphic images.
- (2) $\rho_{\infty} = \mathfrak{R} = \{(M, N) \mid N \leq M, M \in R\text{-Mod}\}$.
- (3) $\rho_{\tau} = \{(M, N) \in \mathfrak{R} \mid N \text{ is } \tau\text{-dense in } M, M \in R\text{-Mod}\}$, where τ is a hereditary torsion theory.
- (4) $\rho_r = \{(M, N) \in \mathfrak{R} \mid N \leq M \text{ such that } r(M/N) = M/N, M \in R\text{-Mod}\}$, where r is a left exact preradical.
- (5) $\rho_{\max} = \{(M, N) \in \mathfrak{R} \mid N \text{ is a maximal submodule in } M \text{ or } N = M, M \in R\text{-Mod}\}$.
- (6) $\rho_e = \{(M, N) \in \mathfrak{R} \mid N \leq_e M, M \in R\text{-Mod}\}$.

It is clear that the P -filters from (2) to (5) are special cases of P -filter in (1). Also, if ρ is a P -filter then the subclass $\rho_R = \{(R, I) \in \rho \mid I \text{ is a left ideal of } R\}$ of \mathfrak{R} is also P -filter.

Throughout this chapter we will fix the following notations.

- For any two P -filters ρ_1 and ρ_2 , we will denote by $\mathcal{L}_{(\rho_1, \rho_2)}$ the subclass $\mathcal{L}_{(\rho_1, \rho_2)} = \{(M, N, f, Q) \in \mathcal{M} \mid M, N, Q \in R\text{-Mod}, (M, N) \in \rho_1 \text{ and } f \in \text{Hom}_R(N, Q) \text{ such that } (M, \ker(f)) \in \rho_2\}$.

- For any two nonempty classes of modules \mathcal{T} and \mathcal{F} , we will denote by $\mathcal{L}_{(\mathcal{T}, \mathcal{F})}$ the subclass $\mathcal{L}_{(\mathcal{T}, \mathcal{F})} = \{(M, N, f, Q) \in \mathcal{M} \mid M, N, Q \in R\text{-Mod}, N \leq M \text{ such that } M/N \in \mathcal{T} \text{ and } f \in \text{Hom}_R(N, Q) \text{ with } M/\ker(f) \in \mathcal{F}\}$. It is clear that $\mathcal{L}_{(\mathcal{T}, \mathcal{F})} = \mathcal{L}_{(\rho_{\mathcal{T}}, \rho_{\mathcal{F}})}$, when \mathcal{T} and \mathcal{F} are closed under submodules and homomorphic images.

- For any two preradicals r and s , we will denote by $\mathcal{L}_{(r, s)}$ the subclass $\mathcal{L}_{(r, s)} = \{(M, N, f, Q) \in \mathcal{M} \mid M, N, Q \in R\text{-Mod}, N \leq M \text{ such that } r(M/N) = M/N \text{ and } f \in \text{Hom}_R(N, Q) \text{ with } s(M/\ker(f)) = M/\ker(f)\}$. It is clear that $\mathcal{L}_{(r, s)} = \mathcal{L}_{(\rho_r, \rho_s)}$, when r and s are left exact preradicals.

- For any torsion theory τ , we will denote by \mathcal{L}_τ the subclass $\mathcal{L}_\tau = \{(M, N, f, Q) \in \mathcal{M} \mid M, N, Q \in R\text{-Mod}, N \text{ is a } \tau\text{-dense in } M \text{ and } f \in \text{Hom}_R(N, Q)\}$. It is clear that $\mathcal{L}_\tau = \mathcal{L}_{(\rho_\tau, \rho_\infty)}$, when τ is a hereditary torsion theory.

Lemma 2.11. *Let ρ_1 and ρ_2 be two P -filters. Then $\mathcal{L}_{(\rho_1, \rho_2)}$ satisfies conditions (α) , (δ) and (μ) .*

Proof. Conditions (α) and (δ) are clear.

Condition (μ) : Let $(M, N, f, Q) \in \mathcal{L}_{(\rho_1, \rho_2)}$ and let $x \in M$, thus $(M, N) \in \rho_1$ and $f \in \text{Hom}_R(N, Q)$ such that $(M, \ker(f)) \in \rho_2$. Since ρ_1 is a P -filter, thus $(R, (N : x)) \in \rho_1$. It is easy to prove that $\ker(f_x) = (\ker(f) : x)$. Since $(M, \ker(f)) \in \rho_2$ and ρ_2 is a P -filter, $(R, (\ker(f) : x)) \in \rho_2$ and hence $(R, \ker(f_x)) \in \rho_2$ and this implies that $(R, (N : x), f_x, Q) \in \mathcal{L}_{(\rho_1, \rho_2)}$. Therefore $\mathcal{L}_{(\rho_1, \rho_2)}$ satisfies condition (μ) . \square

One well-known result concerning injective modules states that an R -module M is injective if and only if every homomorphism from a left ideal of R to M extends to a homomorphism from R to M if and only if for each left ideal I of R and every $f \in \text{Hom}_R(I, M)$, there is an $m \in M$ such that $f(r) = rm, \forall r \in I$. This is known as Baer's condition [4]. Baer's result shows that the left ideals of R form a test set for injectivity.

The following theorem is the main result in this section in which we give a version of Baer's criterion for \mathcal{L} -injectivity.

Theorem 2.12. (Generalized Baer's Criterion) *Consider the following three conditions for an R -module M :*

- (1) M is \mathcal{L} -injective;
- (2) every diagram

$$\begin{array}{ccc} I & \xhookrightarrow{i} & R \\ f \downarrow & & \\ M & & \end{array}$$

with $(R, I, f, M) \in \mathcal{L}$ can be completed to a commutative diagram;

- (3) for each $(R, I, f, M) \in \mathcal{L}$, there exists an element $m \in M$ such that $f(r) = rm, \forall r \in I$.

Then (2) and (3) are equivalent and (1) implies (2). Moreover, if \mathcal{L} satisfies conditions (α) and (μ) , then all the three conditions are equivalent.

Proof. (1) \Rightarrow (2) and (2) \Leftrightarrow (3) are obvious.

(2) \Rightarrow (1) Let \mathcal{L} satisfy conditions (α) and (μ) and consider the following diagram

$$\begin{array}{ccc} A & \xhookrightarrow{i} & B \\ f \downarrow & & \\ M & & \end{array}$$

with $(B, A, f, M) \in \mathcal{L}$. Let $S = \{(C, \varphi) \mid A \leq C \leq B, \varphi \in \text{Hom}_R(C, M) \text{ such that } (\varphi \upharpoonright A) = f\}$. Define on S a partial order \preceq by

$$(C_1, \varphi_1) \preceq (C_2, \varphi_2) \iff C_1 \leq C_2 \text{ and } (\varphi_2 \upharpoonright C_1) = \varphi_1$$

Clearly, $S \neq \emptyset$ since $(A, f) \in S$. Furthermore, one can show that S is inductive in the following manner. Let $F = \{(A_i, f_i) \mid i \in I\}$ be an ascending chain in S . Let $A_\infty = \cup_{i \in I} A_i$. Then for any $a \in A_\infty$ there is a $j \in I$ such that $a \in A_j$, and so we can define $f_\infty : A_\infty \rightarrow M$, by $f_\infty(a) = f_j(a)$. It is straightforward to check that f_∞ is well defined and (A_∞, f_∞) is an upper bound for F in S . Then by Zorn's Lemma, S has a maximal element, say (B', g') . We will prove that $B' = B$.

Suppose that there exists $x \in B \setminus B'$. It is clear that $(B, A, f, M) \preceq (B, B', g', M)$. Since $(B, A, f, M) \in \mathcal{L}$ and \mathcal{L} satisfies condition (α) , thus $(B, B', g', M) \in \mathcal{L}$. Since \mathcal{L} satisfies condition (μ) , thus $(R, (B' : x), g'_x, M) \in \mathcal{L}$. By hypothesis, there exists a homomorphism $g : R \rightarrow M$ such that $g(r) = g'_x(r) = g'(rx), \forall r \in (B' : x)$. Define $\psi : B' + Rx \rightarrow M$ by $\psi(b + rx) = g'(b) + g(r), \forall b \in B', \forall r \in R$. ψ is a well-defined mapping, since if $b_1 + r_1x = b_2 + r_2x$ where $b_1, b_2 \in B'$ and $r_1, r_2 \in R$, then $(r_1 - r_2)x = b_2 - b_1 \in B'$ and hence $r_1 - r_2 \in (B' : x)$. Since $g'((r_1 - r_2)x) = g(r_1 - r_2) = g(r_1) - g(r_2)$ and $g'((r_1 - r_2)x) = g'(b_2 - b_1) = g'(b_2) - g'(b_1)$ thus $g'(b_1) + g(r_1) = g'(b_2) + g(r_2)$ and this implies that $\psi(b_1 + r_1x) = \psi(b_2 + r_2x)$. Thus ψ is a well-defined mapping. It is clear that ψ is a homomorphism

and $(B', g') \preceq (B' + Rx, \psi)$. Since $(B' + Rx, \psi) \in S$ and $B' \subsetneq B' + Rx$, thus we have a contradiction to maximality of (B', g') in S . Hence $B' = B$ and this means that there exists a homomorphism $g' : B \rightarrow M$ such that $(g'|_A) = f$. Thus M is \mathcal{L} -injective. \square

The following corollary is a generalization of Baer's result in [4], [23, Proposition 2.1, p. 201], [16, Baer's Lemma 2.2, p. 628] and [5, Theorem 2.4, p. 319].

Corollary 2.13. *Let p_1 and p_2 be any two P -filters. Then the following conditions are equivalent for R -module M :*

- (1) M is $\mathcal{L}_{(p_1, p_2)}$ -injective;
- (2) every diagram

$$\begin{array}{ccc} I & \xrightarrow{i} & R \\ f \downarrow & & \\ & & M \end{array}$$

with $(R, I, f, M) \in \mathcal{L}_{(p_1, p_2)}$, can be completed to a commutative diagram;

- (3) for each $(R, I, f, M) \in \mathcal{L}_{(p_1, p_2)}$, there exists an element $m \in M$ such that $f(r) = rm, \forall r \in I$.

Proof. By Lemma 2.11 and Theorem 2.12. \square

The following characterization of \mathcal{L} -injectivity is a generalization of [22, Proposition 1.4, p. 3] and [9, Proposition 2.1.3, p. 53].

Proposition 2.14. *Consider the following three conditions for R -module M :*

- (1) Q is \mathcal{L} -injective;
- (2) every diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ f \downarrow & & \\ & & Q \end{array}$$

with $(M, N, f, Q) \in \mathcal{L}$ and $N \leq_e M$, can be completed to a commutative diagram;

- (3) every diagram

$$\begin{array}{ccc} I & \xrightarrow{i} & R \\ f \downarrow & & \\ & & Q \end{array}$$

with $(R, I, f, Q) \in \mathcal{L}$ and $I \leq_e R$, can be completed to a commutative diagram.

Then (1) implies (2), (2) implies (3) and, if \mathcal{L} satisfies conditions (α) and (μ) , then (3) implies (1).

Proof. Let \mathcal{L} satisfy (α) and (μ) and consider the following diagram

$$\begin{array}{ccc} I & \xrightarrow{i} & R \\ f \downarrow & & \\ & & Q \end{array}$$

with $(R, I, f, Q) \in \mathcal{L}$. Let I^c be a complement left ideal of I in R and let $C = I \oplus I^c$. Thus by [2, Proposition 5.21, p. 75], $C \leq_e R$. Define $g : C = I \oplus I^c \rightarrow Q$ by $g(a+b) = f(a)$, $\forall a \in I$ and $\forall b \in I^c$. It is clear that g is a well-defined homomorphism and $(R, I, f, Q) \preceq (R, C, g, Q)$. Since \mathcal{L} satisfies condition (α) , thus $(R, C, g, Q) \in \mathcal{L}$. By hypothesis, there exists a homomorphism $h : R \rightarrow Q$ such that $(h \upharpoonright C) = g$. Thus $(h \upharpoonright I) = (g \upharpoonright I) = f$ and this implies that Q is \mathcal{L} -injective, by Theorem 2.12. \square

In the following theorem we extend a characterization due to [24, Theorem 2, p. 8] of \mathcal{L} -injective modules over commutative Noetherian rings.

Theorem 2.15. Let R be a commutative Noetherian ring, let M be an R -module and suppose that \mathcal{L} satisfies conditions (α) and (μ) . Then M is \mathcal{L} -injective if and only if every diagram

$$\begin{array}{ccc} I & \xhookrightarrow{i} & R \\ f \downarrow & & \\ M & & \end{array}$$

with $(R, I, f, M) \in \mathcal{L}$ and I is a prime ideal of R , can be completed to a commutative diagram.

Proof. (\implies) This is obvious.

(\impliedby) Consider the following diagram

$$\begin{array}{ccc} A & \xhookrightarrow{i} & B \\ f \downarrow & & \\ M & & \end{array}$$

with $(B, A, f, M) \in \mathcal{L}$. Let $S = \{(C, \varphi) \mid A \leq C \leq B, \varphi \in \text{Hom}_R(C, M) \text{ such that } (\varphi \upharpoonright A) = f\}$. Define on S a partial order \preceq by

$$(C_1, \varphi_1) \preceq (C_2, \varphi_2) \iff C_1 \leq C_2 \text{ and } (\varphi_2 \upharpoonright C_1) = \varphi_1$$

As in the proof of Theorem 2.12, we can prove that S has a maximal element, say (B', g') . We will prove that $B' = B$. Suppose that there exists $x \in B \setminus B'$. By [24, Theorem 1, p. 8], there exists an element $r_0 \in R$ such that $(B' : r_0 x)$ is a prime ideal in R and $r_0 x \notin B'$. It is clear that $(B, A, f, M) \preceq (B, B', g', M)$. Since $(B, A, f, M) \in \mathcal{L}$ and \mathcal{L} satisfies condition (α) , thus $(B, B', g', M) \in \mathcal{L}$. Since \mathcal{L} satisfies condition (μ) , thus $(R, (B' : b), g'_b, M) \in \mathcal{L}, \forall b \in B$. Put $y = r_0 x$, thus $y \in B \setminus B'$ and hence $(R, (B' : y), g'_y, M) \in \mathcal{L}$. Thus we have the following diagram

$$\begin{array}{ccc} (B' : y) & \xhookrightarrow{i} & R \\ g'_y \downarrow & \nearrow g & \\ M & & \end{array}$$

with $(R, (B' : y), g'_y, M) \in \mathcal{L}$ and $(B' : y)$ is a prime ideal in R . By hypothesis, there exists a homomorphism $g : R \rightarrow M$ such that $g(r) = g'_y(r) = g'(ry), \forall r \in (B' : y)$. Define $\psi : B' + Ry \rightarrow M$ by $\psi(b + ry) = g'(b) + g(r), \forall b \in B', \forall r \in R$. As in the proof of Theorem 2.12, we can prove that ψ is a well-defined homomorphism and $(B', g') \preceq (B' + Ry, \psi)$. Since $(B' + Ry, \psi) \in S$ and $B' \subsetneq B' + Ry$, thus we have a contradiction to maximality of (B', g') in S . Hence $B' = B$ and this mean that there exists a homomorphism $g' : B \rightarrow M$ such that $(g' \upharpoonright A) = f$. Thus M is \mathcal{L} -injective. \square

Corollary 2.16. Let ρ_1 and ρ_2 be any two P -filters and let R be a commutative Noetherian ring, let M be an R -module. Then M is $\mathcal{L}_{(\rho_1, \rho_2)}$ -injective if and only if every diagram

$$\begin{array}{ccc} I & \xhookrightarrow{i} & R \\ f \downarrow & & \\ M & & \end{array}$$

with $(R, I, f, M) \in \mathcal{L}_{(\rho_1, \rho_2)}$ and I is a prime ideal of R , can be completed to a commutative diagram.

Proof. By Lemma 2.11 and Theorem 2.15. \square

Corollary 2.17. ([24, Theorem 2, p. 8]) Let R be a commutative Noetherian ring, let M be an R -module. Then M is injective if and only if every diagram

$$\begin{array}{ccc} I & \xhookrightarrow{i} & R \\ f \downarrow & & \\ M & & \end{array}$$

with I is a prime ideal of R , can be completed to a commutative diagram.

Proof. By taking the two P -filters $\rho_1 = \rho_2 = \mathfrak{R}$ and applying Corollary 2.16. \square

3 \mathcal{L} - M -Injectivity and s - \mathcal{L} - M -Injectivity

In this section we introduce the concepts of \mathcal{L} - M -injective modules and s - \mathcal{L} - M -injective modules as generalizations of M -injective modules and give some results about them.

Definition 3.1. Let $M, Q \in R\text{-Mod}$. A module Q is said to be \mathcal{L} - M -injective, if every diagram

$$\begin{array}{ccc} N & \xhookrightarrow{i} & M \\ f \downarrow & & \\ Q & & \end{array}$$

with $(M, N, f, Q) \in \mathcal{L}$, can be completed to a commutative diagram. A module Q is said to be \mathcal{L} -quasi-injective, if Q is \mathcal{L} - Q -injective.

Remarks 3.2.

(1) It is clear that an R -module Q is \mathcal{L} -injective if and only if Q is \mathcal{L} - M -injective for all $M \in R\text{-Mod}$.

(2) If \mathcal{L} satisfies conditions (α) and (μ) , then by Theorem 2.12 we have that an R -module Q is \mathcal{L} -injective if and only if Q is \mathcal{L} - R -injective.

(3) As a special case of (2), we have that for any two P -filters ρ_1 and ρ_2 , then an R -module Q is $\mathcal{L}_{(\rho_1, \rho_2)}$ -injective if and only if Q is $\mathcal{L}_{(\rho_1, \rho_2)}$ - R -injective.

For any nonempty subclass \mathcal{L} of \mathcal{M} and for any R -module M , we will denote by \mathcal{L}_M the subclass $\mathcal{L}_M = \{(B, A, f, Q) \in \mathcal{L} \mid B = M\}$.

Remark 3.3. Let $M, Q \in R\text{-Mod}$. Then Q is \mathcal{L} - M -injective if and only if Q is \mathcal{L}_M -injective.

It is clear that if \mathcal{L} satisfies condition (α) , then \mathcal{L}_M does also. Thus, immediately from Remark 3.3 and [16, Theorem 1.3, p. 623], we have the following corollary.

Corollary 3.4. Let $M, Q \in R\text{-Mod}$ and let \mathcal{L} satisfy condition (α) . Then the following five conditions are equivalent:

- (1) Q is \mathcal{L} - M -injective;
- (2) Q is a direct summand in each extension $N \supseteq Q$ such that $N \subseteq Q + r_{\mathcal{L}_M}(E(N), Q)$;
- (3) $r_{\mathcal{L}_M}(Q) \subseteq Q$;
- (4) every diagram

$$\begin{array}{ccc} A & \xhookrightarrow{i} & B \\ f \downarrow & & \\ Q & & \end{array}$$

with $B \subseteq A + r_{\mathcal{L}_M}(E(B), A, f, Q)$, can be completed to a commutative diagram;

- (5) $s_{\mathcal{L}_M}(Q) \subseteq Q$.

Let $M, Q \in R\text{-Mod}$, it is well-known that a module Q is M -injective if and only if $f(M) \leq Q$, for every homomorphism $f : E(M) \rightarrow E(Q)$ [20, Lemma 1.13, p. 7].

For an analogous result for \mathcal{L} - M -injectivity we first fix the following condition.

$(E_{\mathcal{L}})$: Let \mathcal{L} be a subclass of \mathcal{M} . Then a module M satisfies condition $(E_{\mathcal{L}})$ if M has an \mathcal{L} -injective envelope which is unique up to M -isomorphism and $(E_{\mathcal{L}}(M), N, f, Q) \in \mathcal{L}$ whenever $(M, N, f, Q) \in \mathcal{L}$.

The next theorem is the first main result in this section in which we give a generalization of [20, Lemma 1.13, p. 7] and [8, Theorem 2.1, p. 34].

Theorem 3.5. Let $M, Q \in R\text{-Mod}$ and let \mathcal{L} satisfy conditions (α) , (β) and (γ) . Consider the following two conditions

- (1) Q is \mathcal{L} - M -injective.
- (2) $f(M) \leq Q$, for all $f \in \text{Hom}_R(E_{\mathcal{L}}(M), E_{\mathcal{L}}(Q))$ with $(M, L, f \upharpoonright L, Q) \in \mathcal{L}$ where $L = \{m \in M \mid f(m) \in Q\} = M \cap f^{-1}(Q)$.

Then (1) implies (2) and, if M satisfies condition $(E_{\mathcal{L}})$, then (2) implies (1).

Proof. (1) \Rightarrow (2) Let $f \in \text{Hom}_R(E_{\mathcal{L}}(M), E_{\mathcal{L}}(Q))$ with $(M, L, f|_L, Q) \in \mathcal{L}$ where $L = \{m \in M \mid f(m) \in Q\} = M \cap f^{-1}(Q)$. Define $g : L \rightarrow Q$ by $g(a) = f(a)$, $\forall a \in L$. (i.e., $g = f|_L$). It is clear that g is a homomorphism. Thus we have the following diagram

$$\begin{array}{ccc} L & \xrightarrow{i} & M \\ g \downarrow & \swarrow h & \\ Q & & \end{array}$$

with $(M, L, g, Q) \in \mathcal{L}$. By \mathcal{L} - M -injectivity of Q , there exists a homomorphism $h : M \rightarrow Q$ such that $(h|_L) = g$. We will prove that $Q \cap (f-h)(M) = 0$. Let $x \in Q \cap (f-h)(M)$, thus there exists $m \in M$ such that $x = (f-h)(m) = f(m) - h(m) \in Q$. Thus $f(m) = x + h(m) \in Q$ and this implies that $m \in L$. Thus $f(m) = g(m) = h(m)$ and hence $x = 0$. Thus $Q \cap (f-h)(M) = 0$. Since Q is an essential submodule of $E_{\mathcal{L}}(Q)$ (by [16, Theorem 1.19, p. 627]) and since $(f-h)(M) \leq E_{\mathcal{L}}(Q)$, thus $(f-h)(M) = 0$ and this implies that $f(M) = h(M) \leq Q$.

(2) \Rightarrow (1) Let M satisfy condition $(E_{\mathcal{L}})$ and consider the following diagram:

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ f \downarrow & & \\ Q & & \end{array}$$

with $(M, N, f, Q) \in \mathcal{L}$. Since M satisfies condition $(E_{\mathcal{L}})$, thus $(E_{\mathcal{L}}(M), N, f, Q) \in \mathcal{L}$. Since \mathcal{L} satisfies condition (β) , thus $(E_{\mathcal{L}}(M), N, if, E_{\mathcal{L}}(Q)) \in \mathcal{L}$. Thus we have the following diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & E_{\mathcal{L}}(M) \\ f \downarrow & \swarrow h & \\ Q & & \\ i \downarrow & & \\ E_{\mathcal{L}}(Q) & & \end{array}$$

with $(E_{\mathcal{L}}(M), N, if, E_{\mathcal{L}}(Q)) \in \mathcal{L}$. By \mathcal{L} -injectivity of $E_{\mathcal{L}}(Q)$, there exists a homomorphism $h : E_{\mathcal{L}}(M) \rightarrow E_{\mathcal{L}}(Q)$ such that $h(n) = f(n)$, $\forall n \in N$. Let $L = \{m \in M \mid h(m) \in Q\}$. We will prove that $(M, L, g, Q) \in \mathcal{L}$, where $g = h|_L$. Let $x \in N$, thus $h(x) = f(x) \in Q$ and hence $x \in L$. Thus $N \leq L$ and $(g|_N) = f$. Thus $(M, N, f, Q) \preceq (M, L, g, Q)$. Since \mathcal{L} satisfies condition (α) , thus $(M, L, g, Q) \in \mathcal{L}$. By hypothesis, we have that $h(M) \leq Q$ and hence $h' = h|_M : M \rightarrow Q$ is such that $(h'|_N) = f$. Thus Q is an \mathcal{L} - M -injective module. \square

Corollary 3.6. Let $M, Q \in R\text{-Mod}$ and let ρ_1 and ρ_2 be any two P -filters. If M satisfies condition $(E_{\mathcal{L}(\rho_1, \rho_2)})$, then the following two conditions are equivalent.

- (1) Q is $\mathcal{L}(\rho_1, \rho_2)$ - M -injective;
- (2) $f(M) \leq Q$, for all $f \in \text{Hom}_R(E_{\mathcal{L}(\rho_1, \rho_2)}(M), E_{\mathcal{L}(\rho_1, \rho_2)}(Q))$ with $(M, L, f|_L, Q) \in \mathcal{L}$ where $L = \{m \in M \mid f(m) \in Q\} = M \cap f^{-1}(Q)$.

Proof. By Lemma 2.11 and Theorem 3.5. \square

The following lemma is easily proved.

Lemma 3.7. If $\mathcal{L} = \mathcal{M}$, then the following conditions are satisfied.

- (1) A module M is injective if and only if M is \mathcal{L} -injective.
- (2) $E_{\mathcal{L}}(M) = E(M)$, for all $M \in R\text{-Mod}$.
- (3) Every module satisfies condition $(E_{\mathcal{L}})$.

Lemma 3.8. If τ is a hereditary torsion theory, then the following conditions are satisfied.

- (1) A module M is τ -injective if and only if M is \mathcal{L}_{τ} -injective.
- (2) $E_{\mathcal{L}_{\tau}}(M) = E_{\tau}(M)$, for all $M \in R\text{-Mod}$.
- (3) Every module satisfies condition $(E_{\mathcal{L}_{\tau}})$.
- (4) For any $M, Q \in R\text{-Mod}$, if $f \in \text{Hom}_R(E_{\mathcal{L}_{\tau}}(M), E_{\mathcal{L}_{\tau}}(Q))$, then $(M, L, f|_L, Q) \in \mathcal{L}_{\tau}$, where $L = \{m \in M \mid f(m) \in Q\}$.

Proof. (1), (2) and (3) are clear.

(4) Let $M, Q \in R\text{-Mod}$ and let $f \in \text{Hom}_R(E_{\mathcal{L}_\tau}(M), E_{\mathcal{L}_\tau}(Q))$. Let $L = \{m \in M \mid f(m) \in Q\}$. Define $g : (M/L) \rightarrow (E_{\mathcal{L}_\tau}(Q)/Q)$ by $g(m+L) = f(m) + Q$, $\forall m \in M$. It is clear that g is a well-defined R -monomorphism. Since Q is a τ -dense in $E_{\mathcal{L}_\tau}(Q)$, $(E_{\mathcal{L}_\tau}(Q)/Q)$ is a τ -torsion. Since $g(M/L) \leq (E_{\mathcal{L}_\tau}(Q)/Q)$ and $(M/L) \simeq g(M/L)$, thus M/L is a τ -torsion. Hence L is a τ -dense in M . Thus $(M, L, f|_L, Q) \in \mathcal{L}_\tau$. \square

In the special case $\mathcal{L} = \mathcal{M}$ is the result [20, Lemma 1.13, p. 7] mentioned earlier and the following corollary is a generalization of [8, Theorem 2.1, p. 34].

Corollary 3.9. *Let $M, Q \in R\text{-Mod}$ and let τ be any hereditary torsion theory. Then the following conditions are equivalent.*

- (1) Q is τ - M -injective.
- (2) Q is \mathcal{L}_τ - M -injective.
- (3) $f(M) \leq Q$, for every homomorphism $f : E_\tau(M) \rightarrow E_\tau(Q)$.

Proof. By Lemma 2.11, Lemma 3.8 and Theorem 3.5. \square

Let $M, Q \in R\text{-Mod}$ and let τ be any hereditary torsion theory. A module Q is s - τ - M -injective if, for any $N \leq M$, any homomorphism from a τ -dense submodule of N to Q extends to a homomorphism from N to Q [7, Definition 14.6, p. 65].

As a generalization of s - τ - M -injectivity and hence of M -injectivity we introduce the concept of s - \mathcal{L} - M -injectivity as follows.

Definition 3.10. *Let $M, Q \in R\text{-Mod}$. A module Q is said to be s - \mathcal{L} - M -injective if Q is \mathcal{L} - N -injective, for all $N \leq M$. A module Q is said to be s - \mathcal{L} -quasi-injective if Q is s - \mathcal{L} - Q -injective.*

Remarks 3.11.

- (1) *It is clear that every s - \mathcal{L} - M -injective module is \mathcal{L} - M -injective.*
- (2) *Let $M, Q \in R\text{-Mod}$. Then the following conditions are equivalent.*
 - (a) Q is s - \mathcal{L} - M -injective.
 - (b) Q is s - \mathcal{L} - N -injective, for all $N \leq M$.
 - (c) Q is \mathcal{L} - N -injective, for all $N \leq M$.

Fuchs in [14] has obtained a condition similar to Baer's Criterion that characterizes quasi-injective modules, Bland in [6] has generalized that to s - τ -quasi-injective modules, and Charalambides in [7] has generalized that to s - τ - M -injective modules.

Our next aim is to generalize the condition once again in order to characterize s - \mathcal{L} - M -injective modules. We begin with the following condition.

(\mathcal{L}): Let \mathcal{L} be a subclass of \mathcal{M} and let M be a module. Then M satisfies condition (\mathcal{L}) if for every $(B, A, f, Q) \in \mathcal{L}$, then $(Rm, (A : x)m, f_{(x,m)}, Q) \in \mathcal{L}$, for all $m \in M$ and $x \in B$ with $\text{ann}_R(m) \subseteq (\ker(f) : x)$, where $f_{(x,m)} : (A : x)m \rightarrow Q$ is a well-defined homomorphism defined by $f_{(x,m)}(rm) = f(rx)$, for all $r \in (A : x)$.

A subclass \mathcal{L} of \mathcal{M} is said to be fully subclass if every R -module satisfies condition (\mathcal{L}).

Example 3.12. *All of the following subclasses of \mathcal{M} are fully subclasses.*

- (1) $\mathcal{L}_{(T,F)}$, where T and F are nonempty classes of modules closed under submodules and homomorphic images.
- (2) $\mathcal{L} = \mathcal{M}$.
- (3) \mathcal{L}_τ , where τ is a hereditary torsion theory.
- (4) $\mathcal{L}_{(\rho,\sigma)}$, where ρ and σ are left exact preradicals.

Proof. (1) Let $(B, A, f, Q) \in \mathcal{L}_{(T,F)}$ and let $m \in M, x \in B$ such that $\text{ann}_R(m) \subseteq (\ker(f) : x)$. By Lemma 2.11 we have that $\mathcal{L}_{(T,F)}$ satisfies condition (μ) and this implies that $(B, (A : x), f_x, Q) \in \mathcal{L}_{(T,F)}$. It is clear that $(R/(Im : m)) \simeq (Rm/Im)$ and $(R/(Jm : m)) \simeq (Rm/Jm)$ where $I = (A : x)$ and $J = \ker(f_x)$. Since $\mathcal{L}_{(T,F)} = \mathcal{L}_{(\rho_1, \rho_2)}$, where ρ_1 and ρ_2 defined by $\rho_1 = \{(M, N) \in \mathfrak{R} \mid N \leq M \text{ such that } M/N \in T, M \in R\text{-Mod}\}$ and $\rho_2 = \{(M, N) \in \mathfrak{R} \mid N \leq M \text{ such that } M/N \in F, M \in R\text{-Mod}\}$, thus $(R, I) \in \rho_1$ and $(R, J) \in \rho_2$. Since $I \leq (Im : m) \leq R$ and $J \leq (Jm : m) \leq R$ and ρ_1, ρ_2 are P -filters (by example 2.10), thus $(R, (Im : m)) \in \rho_1$ and $(R, (Jm : m)) \in \rho_2$ and this implies that $(R/(Im : m)) \in T$ and $(R/(Jm : m)) \in F$. Since T and F are closed under homomorphic images, thus $(Rm/Im) \in T$ and

$(Rm/Jm) \in F$. Since $\ker(f_x) = \{r \in I \mid f_x(r) = 0\} = \{r \in I \mid f(rx) = 0\} = \{r \in I \mid rx \in \ker(f)\} = \{r \in I \mid r \in (\ker(f) : x)\} = (\ker(f) : x)$ and $\ker(f_{(x,m)}) = \{rm \in Im \mid f_{(x,m)}(rm) = 0\} = \{rm \in Im \mid f(rx) = 0\} = \{rm \in Im \mid rx \in \ker(f)\} = \{rm \in Im \mid r \in (\ker(f) : x)\} = (\ker(f) : x)m$, thus $\ker(f_{(x,m)}) = \ker(f_x)m = Jm$ and this implies that $(Rm/(A : x)m) \in T$ and $(Rm/\ker(f_{(x,m)})) \in F$. Thus $(Rm, (A : x)m, f_{(x,m)}, Q) \in \mathcal{L}_{(T,F)}$ and hence $\mathcal{L}_{(T,F)}$ is a fully subclass.

(2), (3) and (4) are special cases of (1). \square

For any R -module M , we let $\Omega(M)$ denote the set of all left ideals of R which contain the left annihilator of an element of M , (i.e., for any left ideal I of R we have $I \in \Omega(M)$ if and only if there is $m \in M$ such that $\text{ann}_R(m) \subseteq I$).

The following proposition is the second main result in this section, in which we generalize [7, Proposition 14.12, p. 66], [6, Proposition 1, p. 1954] and Fuchs's result in [14], and it is necessary for our version of the Generalized Fuchs Criterion .

Proposition 3.13. *Consider the following statements, where $M, Q \in R\text{-Mod}$:*

- (1) Q is $s\text{-}\mathcal{L}\text{-}M\text{-injective}$;
- (2) if $m \in M$, then every diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & Rm \\ f \downarrow & & \\ Q & & \end{array}$$

with $(Rm, K, f, Q) \in \mathcal{L}$, can be completed to a commutative diagram;

- (3) if $K \leq N$ are modules, not necessarily submodules of M , then every diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & N \\ f \downarrow & & \\ Q & & \end{array}$$

with $(N, K, f, Q) \in \mathcal{L}$ and $\Omega(N) \subseteq \Omega(M)$, can be completed to a commutative diagram.

Then (1) implies (2) and (3) implies (1). Moreover, if \mathcal{L} satisfies condition (α) and M satisfies condition (\mathcal{L}) , then all above statements are equivalent.

Proof. (1) \Rightarrow (2) Let $m \in M$ and consider the following diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & Rm \\ f \downarrow & & \\ Q & & \end{array}$$

with $(Rm, K, f, Q) \in \mathcal{L}$. Since Q is $s\text{-}\mathcal{L}\text{-}M\text{-injective}$, thus Q is $\mathcal{L}\text{-}Rm\text{-injective}$. Thus there exists a homomorphism $g : Rm \rightarrow Q$ such that $(g \upharpoonright K) = f$.

(2) \Rightarrow (3) Let \mathcal{L} satisfy condition (α) and M satisfies condition (\mathcal{L}) and let $K \leq N$ be modules, not necessarily submodules of M . Consider the following diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & N \\ f \downarrow & & \\ Q & & \end{array}$$

with $(N, K, f, Q) \in \mathcal{L}$ and $\Omega(N) \subseteq \Omega(M)$. Let $S = \{(C, \varphi) \mid K \leq C \leq N, \varphi \in \text{Hom}_R(C, M) \text{ such that } (\varphi \upharpoonright K) = f\}$. Define on S a partial order \preceq by

$$(C_1, \varphi_1) \preceq (C_2, \varphi_2) \iff C_1 \leq C_2 \text{ and } (\varphi_2 \upharpoonright C_1) = \varphi_1$$

As in the proof of Theorem 2.12, we can prove that S has a maximal element, say (X, h) . It suffices to show that $X = N$. Suppose that there exists $n \in N \setminus X$. It is clear that $(N, K, f, Q) \preceq (N, X, h, Q)$. Since $(N, K, f, Q) \in \mathcal{L}$ and \mathcal{L} satisfies condition (α) , thus $(N, X, h, Q) \in \mathcal{L}$. Since $\text{ann}_R(n) \in \Omega(N)$ and $\Omega(N) \subseteq \Omega(M)$ (by assumption), thus $\text{ann}_R(n) \in \Omega(M)$ and this implies that there exists $m \in M$

such that $\text{ann}_R(m) \subseteq \text{ann}_R(n)$. Since $\text{ann}_R(n) \subseteq (\ker(h) : n)$, thus $\text{ann}_R(m) \subseteq (\ker(h) : n)$. Since $m \in M$ and $n \in N \setminus X$ such that $\text{ann}_R(m) \subseteq (\ker(h) : n)$ and since M satisfies condition (\mathcal{L}) , thus $(Rm, (X : n)m, h_{(n,m)}, Q) \in \mathcal{L}$. Thus we have the following diagram

$$\begin{array}{ccc} (X : n)m & \xrightarrow{i} & Rm \\ \downarrow h_{(n,m)} & \searrow \varphi^* & \\ Q & & \end{array}$$

with $(Rm, (X : n)m, h_{(n,m)}, Q) \in \mathcal{L}$. By hypothesis, there exists a homomorphism $\varphi^* : Rm \rightarrow Q$ such that $\varphi^*(am) = h_{(n,m)}(am)$, for all $am \in (X : n)m$. Define $h^* : X + Rn \rightarrow Q$ by $h^*(x + rn) = h(x) + \varphi^*(rm)$, $\forall x \in X$ and $\forall r \in R$. h^* is a well-defined mapping, since if $x_1 + r_1n = x_2 + r_2n$, where $x_1, x_2 \in X$ and $r_1, r_2 \in R$, thus $x_2 - x_1 = r_1n - r_2n = (r_1 - r_2)n \in X$ and hence $r_1 - r_2 \in (X : n)$. Thus $(r_1 - r_2)m \in (X : n)m$ and $\varphi^*((r_1 - r_2)m) = h_{(n,m)}((r_1 - r_2)m) = h((r_1 - r_2)n) = h(x_2 - x_1) = h(x_2) - h(x_1)$. Since $\varphi^*((r_1 - r_2)m) = \varphi^*(r_1m - r_2m) = \varphi^*(r_1m) - \varphi^*(r_2m)$, thus $\varphi^*(r_1m) - \varphi^*(r_2m) = h(x_2) - h(x_1)$ and this implies that $\varphi^*(r_1m) + h(x_1) = \varphi^*(r_2m) + h(x_2)$. Thus $h^*(x_1 + r_1n) = h^*(x_2 + r_2n)$ and hence h^* is a well-defined mapping and it is easy to prove that h^* is a homomorphism. For all $a \in K$, we have that $h^*(a) = h^*(a + 0.n) = h(a) + \varphi^*(0.m) = h(a) = f(a)$ and hence $(h^* \upharpoonright K) = f$. Since $K \leq X + Rn \leq N$, thus $(X + Rn, h^*) \in S$. Since $(h^* \upharpoonright X) = h$ and $X \leq X + Rn \leq N$, thus $(X, h) \preceq (X + Rn, h^*)$. Since $n \in X + Rn$ and $n \notin X$, thus $X \subsetneq X + Rn$ and this contradicts the maximality of (X, h) in S . Thus $X = N$ and this implies that there exists a homomorphism $h : N \rightarrow Q$ such that $(h \upharpoonright K) = f$.

(3) \Rightarrow (1) Let $N \leq M$ and consider the following diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & N \\ \downarrow f & & \\ Q & & \end{array}$$

with $(N, K, f, Q) \in \mathcal{L}$. Let $I \in \Omega(N)$, thus there exists an element $n \in N$ such that $\text{ann}_R(n) \subseteq I$. Thus there exists an element $n \in M$ such that $\text{ann}_R(n) \subseteq I$ and this implies that $I \in \Omega(M)$ and hence we have that $\Omega(N) \subseteq \Omega(M)$. By hypothesis, there exists a homomorphism $g : N \rightarrow Q$ such that $(g \upharpoonright K) = f$. Thus Q is \mathcal{L} - N -injective module, for all $N \leq M$ and this implies that Q is $s\mathcal{L}$ - M -injective. \square

Follow we give the last main result in this section in which we generalize [7, Proposition 14.13, p. 68], [6, Proposition 2, p. 1955] and [14, Lemma 2, p. 542]. It is our version of Generalized Fuchs Criterion.

Proposition 3.14. (Generalized Fuchs Criterion) Consider the following conditions, where $M, Q \in R\text{-Mod}$.

- (1) Q is $s\mathcal{L}$ - M -injective;
- (2) every diagram

$$\begin{array}{ccc} I & \xrightarrow{i} & R \\ \downarrow f & & \\ Q & & \end{array}$$

with $(R, I, f, Q) \in \mathcal{L}$ and $\ker(f) \in \Omega(M)$, can be completed to a commutative diagram;

- (3) for each $(R, I, f, Q) \in \mathcal{L}$ with $\ker(f) \in \Omega(M)$, there exists an element $x \in Q$ such that $f(a) = ax$, $\forall a \in I$.

Then (2) \Leftrightarrow (3) and if M satisfies condition (\mathcal{L}) then (1) implies (2). Moreover, if \mathcal{L} satisfies conditions (α) and (μ) , then (2) implies (1).

Proof. (2) \Leftrightarrow (3) This is obvious.

- (1) \Rightarrow (2) Let M satisfy condition (\mathcal{L}) and consider the following diagram

$$\begin{array}{ccc} I & \xrightarrow{i_1} & R \\ \downarrow f & & \\ Q & & \end{array}$$

with $(R, I, f, Q) \in \mathcal{L}$ and $\ker(f) \in \Omega(M)$. Thus there exists an element $m \in M$ such that $\text{ann}_R(m) \subseteq \ker(f)$. Since $\ker(f) = (\ker(f) : 1)$ where 1 is the identity element of R , thus $\text{ann}_R(m) \subseteq (\ker(f) : 1)$. Since $1 \in R$ and M satisfies condition (\mathcal{L}) , thus $(Rm, (I : 1)m, f_{(1,m)}, Q) \in \mathcal{L}$. Since $(I : 1) = I$, thus $(Rm, Im, f_{(1,m)}, Q) \in \mathcal{L}$. Thus we have the following diagram

$$\begin{array}{ccc} Im & \xrightarrow{i_2} & Rm \\ f_{(1,m)} \downarrow & \nearrow h & \\ Q & & \end{array}$$

with $(Rm, Im, f_{(1,m)}, Q) \in \mathcal{L}$. Since Q is $s\text{-}\mathcal{L}\text{-}M$ -injective, thus by Proposition 3.13, there exists a homomorphism $h : Rm \rightarrow Q$ such that $h \circ i_2 = f_{(1,m)}$. Define $v_1 : I \rightarrow Im$ by $v_1(a) = am$, $\forall a \in I$ and define $v_2 : R \rightarrow Rm$ by $v_2(r) = rm$, $\forall r \in R$. It is clear that v_1 and v_2 are homomorphisms and for all $a \in I$, we have that $(v_2 \circ i_1)(a) = (v_2(i_1(a))) = v_2(a) = am = v_1(a) = i_2(v_1(a)) = (i_2 \circ v_1)(a)$. Define $g : R \rightarrow Q$ by $g(r) = (h \circ v_2)(r)$, $\forall r \in R$. It is clear that g is a homomorphism and for all $a \in I$ we have that $(g \circ i_1)(a) = (g(i_1(a))) = (h \circ v_2)(i_1(a)) = h((v_2 \circ i_1)(a)) = h((i_2 \circ v_1)(a)) = (h \circ i_2)(v_1(a)) = f_{(1,m)}(v_1(a)) = f_{(1,m)}(am) = f(a.1) = f(a)$. Thus there exists a homomorphism $g : R \rightarrow Q$ such that $(g|I) = f$.

(2) \Rightarrow (1) Let \mathcal{L} satisfy conditions (α) and (μ) and let $N \leq M$. Consider the following diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & N \\ f \downarrow & & \\ Q & & \end{array}$$

with $(N, K, f, Q) \in \mathcal{L}$. Let $S = \{(C, \varphi) \mid K \leq C \leq N, \varphi \in \text{Hom}_R(C, M) \text{ such that } (\varphi|K) = f\}$. Define on S a partial order \preceq by

$$(C_1, \varphi_1) \preceq (C_2, \varphi_2) \iff C_1 \leq C_2 \text{ and } (\varphi_2|C_1) = \varphi_1$$

As in the proof of Theorem 2.12, we can prove that S has a maximal element, say (X, h) . It suffices to show that $X = N$. Suppose that there exists $n \in N \setminus X$. It is clear that $(N, K, f, Q) \preceq (N, X, h, Q)$. Since $(N, K, f, Q) \in \mathcal{L}$ and \mathcal{L} satisfies condition (α) , thus $(N, X, h, Q) \in \mathcal{L}$. Since \mathcal{L} satisfies condition (μ) and $n \in N \setminus X$, thus $(R, (X : n), h_n, Q) \in \mathcal{L}$. Since $(0 : n) \subseteq \ker(h_n)$ and $n \in M$, thus $\ker(h_n) \in \Omega(M)$. Thus we have the following diagram

$$\begin{array}{ccc} (X : n) & \xrightarrow{i} & R \\ h_n \downarrow & \nearrow \varphi^* & \\ Q & & \end{array}$$

with $(R, (X : n), h_n, Q) \in \mathcal{L}$ and $\ker(h_n) \in \Omega(M)$. By hypothesis, there exists a homomorphism $\varphi^* : R \rightarrow Q$ such that $(\varphi^*|(X : n)) = h_n$. Define $h^* : X + Rn \rightarrow Q$ by $h^*(x + rn) = h(x) + \varphi^*(r)$, $\forall x \in X, \forall r \in R$. As in the proof of Theorem 2.12, we can prove that h^* is a well-defined homomorphism, $(X, h) \preceq (X + Rn, h^*)$ and $(X + Rn, h^*) \in S$. Since $n \in X + Rn$ and $n \notin X$, thus $X \subsetneq X + Rn$ and this contradicts the maximality of (X, h) in S . Thus $X = N$ and this implies that there exists a homomorphism $h : N \rightarrow Q$ such that $(h|K) = f$. Thus Q is $\mathcal{L}\text{-}M$ -injective module, for all $N \leq M$. Therefore Q is $s\text{-}\mathcal{L}\text{-}M$ -injective R -module. \square

4 Direct Sums of \mathcal{L} -Injective Modules

In Example 2.5 we showed that a direct sums of \mathcal{L} -injective modules is in general not \mathcal{L} -injective. In this section we study conditions under which the class of \mathcal{L} -injective modules is closed under direct sums.

Let $\{E_\alpha\}_{\alpha \in A}$ be a family of modules and let $E = \bigoplus_{\alpha \in A} E_\alpha$. For any $x = (x_\alpha)_{\alpha \in A} \in E$, we define the support of x to be the set $\{\alpha \in A \mid x_\alpha \neq 0\}$ and denote it by $\text{supp}(x)$. For any $X \subseteq E$, we define $\text{supp}(X)$ to be the set $\bigcup_{x \in X} \text{supp}(x) = \{\alpha \in A \mid (\exists x \in X) x_\alpha \neq 0\}$.

The following condition will be useful later.

(F): Let $\{E_\alpha\}_{\alpha \in A}$ be a family of modules, where A is an infinite index set and let \mathcal{L} be a subclass of \mathcal{M} . We say that \mathcal{L} satisfies condition (F) for a family $\{E_\alpha\}_{\alpha \in A}$, if for any $(R, I, f, \bigoplus_{\alpha \in A} E_\alpha) \in \mathcal{L}$, then $\text{supp}(\text{im}(f))$ is finite.

Lemma 4.1. *Let A be any index set and let C be any countable subset of A , and let $\{E_\alpha\}_{\alpha \in A}$ be any family of modules. Define $\pi_C : \bigoplus_{\alpha \in A} E_\alpha \rightarrow \bigoplus_{\alpha \in C} E_\alpha$ by $\pi_C(x) = x_C$, for all $x \in \bigoplus_{\alpha \in A} E_\alpha$ where $\pi_\alpha(x_C) = \pi_\alpha(\pi_C(x)) = \begin{cases} \pi_\alpha(x) & \text{if } \alpha \in C \\ 0 & \text{if } \alpha \notin C \end{cases}$, $\forall \alpha \in A$, where π_α is the α th projection homomorphism. Then π_C is a well-defined homomorphism and if $x \in \bigoplus_{\alpha \in C} E_\alpha$, then $\pi_C(x) = x$.*

Proof. An easy check. \square

Lemma 4.2. *Let \mathcal{L} satisfy the conditions (α) , (μ) and (δ) and let $\{E_\alpha\}_{\alpha \in A}$ be any family of \mathcal{L} -injective modules, where A is an infinite index set. If \mathcal{L} satisfies condition (F) for a family $\{E_\alpha\}_{\alpha \in A}$, then $\bigoplus_{\alpha \in A} E_\alpha$ is an \mathcal{L} -injective module.*

Proof. Consider the following diagram

$$\begin{array}{ccc} I & \xrightarrow{i} & R \\ f \downarrow & & \\ \bigoplus_{\alpha \in A} E_\alpha & & \end{array}$$

with $(R, I, f, \bigoplus_{\alpha \in A} E_\alpha) \in \mathcal{L}$. Since \mathcal{L} satisfies condition (F) for the family $\{E_\alpha\}_{\alpha \in A}$, thus $\text{supp}(\text{im}(f))$ is finite and this implies that $f(I) \subseteq \bigoplus_{\alpha \in F} E_\alpha$, where F is a finite subset of A . Since E_α is \mathcal{L} -injective, $\forall \alpha \in F$, thus by Corollary 2.7 we have that $\bigoplus_{\alpha \in F} E_\alpha$ is \mathcal{L} -injective. Define $\pi_F : \bigoplus_{\alpha \in A} E_\alpha \rightarrow \bigoplus_{\alpha \in F} E_\alpha$ by $\pi_F(x) = x_F$, for all $x \in \bigoplus_{\alpha \in A} E_\alpha$, where $\pi_\alpha(x_F) = \pi_\alpha(\pi_F(x)) = \begin{cases} \pi_\alpha(x) & \text{if } \alpha \in F \\ 0 & \text{if } \alpha \notin F \end{cases}$, $\forall \alpha \in A$, where π_α is the α th projection homomorphism. By Lemma 4.1, we have that π_F is a well-defined homomorphism. Since $(R, I, f, \bigoplus_{\alpha \in A} E_\alpha) \in \mathcal{L}$ and \mathcal{L} satisfies condition (δ) , thus $(R, I, \pi_F \circ f, \bigoplus_{\alpha \in F} E_\alpha) \in \mathcal{L}$. Thus we have the following diagram

$$\begin{array}{ccc} I & \xrightarrow{i} & R \\ f \downarrow & & \nearrow g \\ \bigoplus_{\alpha \in A} E_\alpha & & \\ \pi_F \downarrow & & \\ \bigoplus_{\alpha \in F} E_\alpha & & \end{array}$$

with $(R, I, \pi_F \circ f, \bigoplus_{\alpha \in F} E_\alpha) \in \mathcal{L}$. By \mathcal{L} -injectivity of $\bigoplus_{\alpha \in F} E_\alpha$, there exists a homomorphism $g : R \rightarrow \bigoplus_{\alpha \in F} E_\alpha$ such that $g(a) = (\pi_F \circ f)(a)$, $\forall a \in I$. Put $g' = i_1 \circ g : R \rightarrow \bigoplus_{\alpha \in A} E_\alpha$, where $i_1 : \bigoplus_{\alpha \in F} E_\alpha \rightarrow \bigoplus_{\alpha \in A} E_\alpha$ is the inclusion homomorphism. Then for each $a \in I$, we have that $g'(a) = (i_1 \circ g)(a) = g(a) = (\pi_F \circ f)(a) = \pi_F(f(a))$. Since $f(I) \subseteq \bigoplus_{\alpha \in F} E_\alpha$, thus $f(a) \in \bigoplus_{\alpha \in F} E_\alpha$, $\forall a \in I$. Thus by Lemma 4.1 we have that $\pi_F(f(a)) = f(a)$, $\forall a \in I$ and this implies that $g'(a) = f(a)$, $\forall a \in I$. Since \mathcal{L} satisfies conditions (α) and (μ) , thus $\bigoplus_{\alpha \in A} E_\alpha$ is \mathcal{L} -injective, by Theorem 2.12. \square

The following proposition generalizes a result found in [15, Proposition 8.13, p. 83].

Proposition 4.3. *Let \mathcal{L} satisfy conditions (α) , (μ) and (δ) and let $\{E_\alpha\}_{\alpha \in A}$ be any family of \mathcal{L} -injective modules, where A is an infinite index set. If $\bigoplus_{\alpha \in C} E_\alpha$ is an \mathcal{L} -injective module for any countable subset C of A , then $\bigoplus_{\alpha \in A} E_\alpha$ is an \mathcal{L} -injective module.*

Proof. It is clear that E_α must be \mathcal{L} -injective, $\forall \alpha \in A$. Let $\pi_\beta : \bigoplus_{\alpha \in A} E_\alpha \rightarrow E_\beta$ be the natural projection homomorphism. Assume that $\bigoplus_{\alpha \in A} E_\alpha$ is not \mathcal{L} -injective, thus by Lemma 4.2 there exists $(R, I, f, \bigoplus_{\alpha \in A} E_\alpha) \in \mathcal{L}$ such that $\text{supp}(\text{im}(f))$ is infinite. Since $\text{supp}(\text{im}(f))$ is an infinite set, thus $\text{supp}(\text{im}(f))$ contains a countable infinite subset, say C . For any $\alpha \in C$, then $\alpha \in \text{supp}(\text{im}(f))$ and this implies that there exists $x \in \text{im}(f)$ such that $x_\alpha \neq 0$. Thus for any $\alpha \in C$, then $\pi_\alpha(\text{im}(f)) \neq$

0. Define $\pi_C : \bigoplus_{\alpha \in A} E_\alpha \rightarrow \bigoplus_{\alpha \in C} E_\alpha$ as in Lemma 4.1. Note that $C = \text{supp}(\text{im}(\pi_C \circ f))$. Since $(R, I, f, \bigoplus_{\alpha \in A} E_\alpha) \in \mathcal{L}$ and \mathcal{L} satisfies condition (γ) , thus $(R, I, \pi_C \circ f, \bigoplus_{\alpha \in C} E_\alpha) \in \mathcal{L}$. Since C is a countable subset of A , thus by hypothesis we have that $\bigoplus_{\alpha \in C} E_\alpha$ is \mathcal{L} -injective. By Theorem 2.12, there exists an element $y \in \bigoplus_{\alpha \in C} E_\alpha$ such that $(\pi_C \circ f)(a) = ay, \forall a \in I$. Let $\alpha \in \text{supp}(\text{im}(\pi_C \circ f))$, thus there is $r \in I$ such that $\pi_\alpha((\pi_C \circ f)(r)) \neq 0$. Hence $\pi_\alpha(ry) \neq 0$ and this implies that $\pi_\alpha(y) \neq 0$. Thus $\alpha \in \text{supp}(y)$ and hence $\text{supp}(\text{im}(\pi_C \circ f)) \subseteq \text{supp}(y)$. Since $C = \text{supp}(\text{im}(\pi_C \circ f))$, thus $C \subseteq \text{supp}(y)$ and this a contradiction since $\text{supp}(y)$ is finite (because $y \in \bigoplus_{\alpha \in C} E_\alpha$) and C is infinite. Thus $\bigoplus_{\alpha \in A} E_\alpha$ is an \mathcal{L} -injective module. \square

By Proposition 4.3 and Lemma 2.11 we can prove the following corollary.

Corollary 4.4. *Let ρ_1 and ρ_2 be any two P -filters and let $\{E_\alpha\}_{\alpha \in A}$ be any family of modules, where A is an infinite index set. If $\bigoplus_{\alpha \in C} E_\alpha$ is an $\mathcal{L}_{(\rho_1, \rho_2)}$ -injective module for any countable subset C of A , then $\bigoplus_{\alpha \in A} E_\alpha$ is an $\mathcal{L}_{(\rho_1, \rho_2)}$ -injective module.*

We can now state the following result, found in [15, Proposition 8.13, p. 83] as a corollary.

Corollary 4.5. *Let $\{E_\alpha\}_{\alpha \in A}$ be any family of τ -injective modules, where A is an infinite index set. If $\bigoplus_{\alpha \in C} E_\alpha$ is a τ -injective module for any countable subset C of A , then $\bigoplus_{\alpha \in A} E_\alpha$ is a τ -injective module.*

Proof. By taking the two P -filters $\rho_1 = \rho_\tau$ and $\rho_2 = \mathfrak{R}$ and applying Corollary 4.4. \square

By Proposition 4.3 and Remark 2.1 we have the following corollary.

Corollary 4.6. *Consider the following three conditions, where \mathcal{K} is a nonempty class of R -modules.*

- (1) *Every direct sum of \mathcal{L} -injective R -modules in \mathcal{K} is \mathcal{L} -injective.*
- (2) *Every countable direct sum of \mathcal{L} -injective R -modules in \mathcal{K} is \mathcal{L} -injective.*
- (3) *For any family $\{E_i\}_{i \in \mathbb{N}}$ of \mathcal{L} -injective R -modules in \mathcal{K} , then $\bigoplus_{i \in \mathbb{N}} E_i$ is \mathcal{L} -injective.*

Then (1) implies (2) and (2) implies (3), and if \mathcal{L} satisfies conditions (α) , (μ) and (δ) , then (2) implies (1). Moreover, if \mathcal{L} satisfies condition (γ) , then (3) implies (2).

Definition 4.7. A submodule N of a module M is said to be strongly \mathcal{L} -dense in M (shortly, $s\text{-}\mathcal{L}$ -dense), if $(M, N, I_N, N) \in \mathcal{L}$, where I_N is the identity homomorphism from N into N .

Lemma 4.8. *The following statements are hold.*

- (1) *Every $s\text{-}\mathcal{L}$ -dense submodule in M is \mathcal{L} -dense in M*
- (2) *If $N \leq K \leq M$ are modules such that N is $s\text{-}\mathcal{L}$ -dense in M and \mathcal{L} satisfies conditions (α) and (β) , then K is $s\text{-}\mathcal{L}$ -dense in M .*

Proof. (1) Let N be any $s\text{-}\mathcal{L}$ -dense submodule in M , thus $(M, N, I_N, N) \in \mathcal{L}$. Since the following diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ I_N \downarrow & & \downarrow i \\ N & \xrightarrow{i} & E(M) \\ I_N \downarrow & & \\ N & & \end{array}$$

is commutative and $(M, N, I_N, N) \in \mathcal{L}$, thus $i(M) \leq r_{\mathcal{L}}(E(M), N)$ and this implies that $M \subseteq r_{\mathcal{L}}(E(M), N) + N$. Thus N is \mathcal{L} -dense in M .

(2) Since N is $s\text{-}\mathcal{L}$ -dense in M , thus $(M, N, I_N, N) \in \mathcal{L}$. Since \mathcal{L} satisfies condition (β) , thus $(M, N, iI_N, K) \in \mathcal{L}$ where $i : N \rightarrow K$ is the inclusion homomorphism and this implies that $(M, N, i, K) \in \mathcal{L}$. Since $N \leq K \leq M$ and $(I_K|N) = i$, thus $(M, N, i, K) \preceq (M, K, I_K, K)$. Since \mathcal{L} satisfies condition (α) , thus $(M, K, I_K, K) \in \mathcal{L}$. Hence K is $s\text{-}\mathcal{L}$ -dense in M . \square

Lemma 4.9. *Let ρ be any P -filter. Then $(M, N) \in \rho$ if and only if N is $s\text{-}\mathcal{L}_{(\rho, \infty)}$ -dense in M .*

Proof. This is obvious. \square

Following [11, p. 21], for any module M , denote by $H_{\mathcal{K}}(R)$ the set of left submodules N of M such that $(M/N) \in \mathcal{K}$, where \mathcal{K} is any nonempty class of modules (i.e., $H_{\mathcal{K}}(M) = \{N \leq M \mid (M/N) \in \mathcal{K}\}$). In particular, $H_{\mathcal{K}}(R) = \{I \leq R \mid (R/I) \in \mathcal{K}\}$.

The following theorem is the first main result in this section.

Theorem 4.10. Let \mathcal{L} satisfy conditions (α) and (δ) and let \mathcal{K} be any nonempty class of modules closed under isomorphic copies and \mathcal{L} -injective hulls. If the direct sum of any family $\{E_i\}_{i \in \mathbb{N}}$ of \mathcal{L} -injective R -modules in \mathcal{K} is \mathcal{L} -injective, then every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R in $H_{\mathcal{K}}(R)$ with $I_{\infty} = \bigcup_{j=1}^{\infty} I_j$ s - \mathcal{L} -dense in R , terminates.

Proof. Let $I_1 \subseteq I_2 \subseteq \dots$ be any ascending chain of left ideals of R in $H_{\mathcal{K}}(R)$ with $I_{\infty} = \bigcup_{j=1}^{\infty} I_j$ being a s - \mathcal{L} -dense left ideal in R . Thus $(R/I_j) \in \mathcal{K}$, $\forall j \in \mathbb{N}$. Since \mathcal{L} satisfies conditions (α) , (β) and (γ) , by Theorem 2.2 we have that every R -module M has an \mathcal{L} -injective hull which is unique up to M -isomorphism. Let $E_{\mathcal{L}}(R/I_j)$ be the \mathcal{L} -injective hull of R/I_j , $\forall j \in \mathbb{N}$. Since \mathcal{K} closed under \mathcal{L} -injective hulls, $E_{\mathcal{L}}(R/I_j) \in \mathcal{K}$, $\forall j \in \mathbb{N}$. Define $f : I_{\infty} = \bigcup_{j=1}^{\infty} I_j \rightarrow \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$ by $f(r) = (r + I_j)_{j \in \mathbb{N}}$, for $r \in I_{\infty}$. Note that f is a well-defined mapping: for any $r \in I_{\infty}$, let n be the smallest positive integer such that $r \in I_n$. Since $I_n \subseteq I_{n+k}$, $\forall k \in \mathbb{N}$, thus $r \in I_{n+k}$, $\forall k \in \mathbb{N}$ and so $r + I_{n+k} = 0$, $\forall k \in \mathbb{N}$. Thus $(r + I_j)_{j \in \mathbb{N}} = (r + I_1, r + I_2, \dots, r + I_{n-1}, 0, 0, \dots) \in \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$. Thus $f(I) \subseteq \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$ and hence f is a well-defined mapping. It is clear that f is a homomorphism. Since I_{∞} is a s - \mathcal{L} -dense left ideal in R , thus $(R, I_{\infty}, I_{\infty}, I_{\infty}) \in \mathcal{L}$. Since \mathcal{L} satisfies condition (δ) , thus $(R, I_{\infty}, f, \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)) \in \mathcal{L}$. Since $E_{\mathcal{L}}(R/I_j)$ is an \mathcal{L} -injective R -module in \mathcal{K} , $\forall j \in \mathbb{N}$, thus by hypothesis we have that $\bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$ is an \mathcal{L} -injective R -module. Thus by Theorem 2.12, there exists an element $x \in \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$ such that $f(r) = rx$, $\forall r \in I_{\infty}$. Since $x \in \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$, thus $x = (x_1, x_2, \dots, x_n, 0, 0, \dots)$, for some $n \in \mathbb{N}$. Hence $(r + I_j)_{j \in \mathbb{N}} = (rx_1, rx_2, \dots, rx_n, 0, 0, \dots)$ and this implies that $r + I_{n+k} = 0$, $\forall k \geq 1$ and $\forall r \in I_{\infty}$, thus $r \in I_{n+k}$, $\forall k \geq 1$ and $\forall r \in I_{\infty}$. Thus $I_{\infty} = \bigcup_{j=1}^{\infty} I_j \subseteq I_{n+k}$, $\forall k \geq 1$. Since $I_{n+k} \subseteq I_{\infty}$, $I_{\infty} = I_{n+k}$, $\forall k \geq 1$, thus $I_t = I_{t+j}$, $\forall j \in \mathbb{N}$. Therefore the ascending chain $I_1 \subseteq I_2 \subseteq \dots$ terminates. \square

Now we will state the condition (I) on \mathcal{L} as follows.

$(I) : (R, J, f, Q) \in \mathcal{L}$ implies that J is s - \mathcal{L} -dense in R . That is, $(R, J, f, Q) \in \mathcal{L}$ implies $(R, J, I_J, J) \in \mathcal{L}$.

Proposition 4.11. Consider the following two conditions, where \mathcal{K} is a nonempty class of R -modules.

- (1) Every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R in $H_{\mathcal{K}}(R)$ with $I_{\infty} = \bigcup_{j=1}^{\infty} I_j$ s - \mathcal{L} -dense in R , terminates.
- (2) The following conditions hold:
 - (a) $H_{\mathcal{K}}(R)$ has ACC on s - \mathcal{L} -dense left ideals in R ;
 - (b) for every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R in $H_{\mathcal{K}}(R)$ with $I_{\infty} = \bigcup_{j=1}^{\infty} I_j$ s - \mathcal{L} -dense in R , there exists a positive integer n such that I_n is s - \mathcal{L} -dense in R .

If \mathcal{L} satisfies conditions (α) and (β) , then (1) and (2) are equivalent.

Proof. Clearly (1) \Rightarrow (2b).

(1) \Rightarrow (2a) Assume that \mathcal{L} satisfies conditions (α) and (β) and let $I_1 \subseteq I_2 \subseteq \dots$ be any ascending chain of s - \mathcal{L} -dense left ideals of R in $H_{\mathcal{K}}(R)$. Since $I_1 \subseteq I_{\infty} = \bigcup_{j=1}^{\infty} I_j$, thus by Lemma 4.8 we have that I_{∞} is s - \mathcal{L} -dense in R . By hypothesis, the chain $I_1 \subseteq I_2 \subseteq \dots$ terminates. Thus $H_{\mathcal{K}}(R)$ has ACC on s - \mathcal{L} -dense left ideals in R .

(2) \Rightarrow (1) Assume that \mathcal{L} satisfies conditions (α) and (β) and let $I_1 \subseteq I_2 \subseteq \dots$ be any ascending chain of left ideals of R in $H_{\mathcal{K}}(R)$, with $I_{\infty} = \bigcup_{j=1}^{\infty} I_j$ s - \mathcal{L} -dense in R . By (2b), there exists a positive integer n such that I_n is s - \mathcal{L} -dense in R . Consider the following ascending chain $I_n \subseteq I_{n+1} \subseteq \dots$ of left ideals of R . Since \mathcal{L} satisfies conditions (α) and (β) , thus by Lemma 4.8 we have that I_{n+j} is s - \mathcal{L} -dense left ideal in R , $\forall j \in \mathbb{N}$. By (2a), there exists a positive integer $t \geq n$ such that $I_t = I_{t+j}$, $\forall j \in \mathbb{N}$. Thus the ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals terminates. \square

Now we will give the second main result in this section.

Theorem 4.12. Let \mathcal{L} satisfy conditions (α) , (μ) , (δ) and (I) and let \mathcal{K} be any nonempty class of modules closed under isomorphic copies and submodules. If every ascending chain $J_1 \subseteq J_2 \subseteq \dots$ of left ideals of R such that $(J_{i+1}/J_i) \in \mathcal{K}$, $\forall i \in \mathbb{N}$ and $J_{\infty} = \bigcup_{i=1}^{\infty} J_i$ s - \mathcal{L} -dense in R terminates, then every direct sum of \mathcal{L} -injective modules in \mathcal{K} is \mathcal{L} -injective.

Proof. Let $\{E_i\}_{i \in \mathbb{N}}$ be any family of \mathcal{L} -injective modules in \mathcal{K} and let $(R, J, f, \bigoplus_{i \in \mathbb{N}} E_i) \in \mathcal{L}$. For any $n \in \mathbb{N}$, put $J_n = \{x \in J \mid f(x) \in \bigoplus_{i=1}^n E_i\} = f^{-1}(\bigoplus_{i=1}^n E_i)$. It is clear that $J_1 \subseteq J_2 \subseteq \dots$. Also, we have that $J_{\infty} = \bigcup_{n \in \mathbb{N}} J_n = \bigcup_{n \in \mathbb{N}} (f^{-1}(\bigoplus_{i=1}^n E_i)) = f^{-1}(\bigcup_{n \in \mathbb{N}} (\bigoplus_{i=1}^n E_i)) = f^{-1}(\bigoplus_{i=1}^{\infty} E_i)$. Since $(R, J, f, \bigoplus_{i \in \mathbb{N}} E_i) \in \mathcal{L}$ and \mathcal{L} satisfies condition (I) , $J = \bigcup_{i \in \mathbb{N}} J_i$ is s - \mathcal{L} -dense in R . For all $n \in \mathbb{N}$, define $\alpha_n : J_{n+1}/J_n \rightarrow \bigoplus_{i=1}^{n+1} E_i / \bigoplus_{i=1}^n E_i$ by $\alpha_n(x + J_n) = f(x) + (\bigoplus_{i=1}^n E_i)$, $\forall x \in J_{n+1}$. α_n is a well-defined mapping and injective, $\forall n \in \mathbb{N}$, since $J_n = f^{-1}(\bigoplus_{i=1}^n E_i)$. It is clear that α_n is homomorphism, $\forall n \in \mathbb{N}$. Since $(\bigoplus_{i=1}^{n+1} E_i / \bigoplus_{i=1}^n E_i) \simeq E_{n+1} \in \mathcal{K}$ and \mathcal{K} closed under isomorphic

copies, thus $(\bigoplus_{i=1}^{n+1} E_i / \bigoplus_{i=1}^n E_i) \in \mathcal{K}$, $\forall n \in \mathbb{N}$. Since $\text{im}(\alpha_n) \leq (\bigoplus_{i=1}^{n+1} E_i / \bigoplus_{i=1}^n E_i) \in \mathcal{K}$, $\forall n \in \mathbb{N}$ and \mathcal{K} closed under submodules, thus $\text{im}(\alpha_n) \in \mathcal{K}$, $\forall n \in \mathbb{N}$. Since $(J_{n+1}/J_n) \simeq \text{im}(\alpha_n)$, $\forall n \in \mathbb{N}$ and \mathcal{K} closed under isomorphic copies, thus $(J_{n+1}/J_n) \in \mathcal{K}$, $\forall n \in \mathbb{N}$. Thus we have the following ascending chain $J_1 \subseteq J_2 \subseteq \dots$ of left ideals of R such that $(J_{i+1}/J_i) \in \mathcal{K}$, $\forall i \in \mathbb{N}$ and $J_\infty = \bigcup_{i=1}^\infty J_i$ is $s\text{-}\mathcal{L}$ -dense in R . By hypothesis, there exists a positive integer n such that $J_n = J_{n+i}$, $\forall i \in \mathbb{N}$. Thus $J = J_\infty = \bigcup_{i=1}^\infty J_i = J_n$. This implies that $f(J) \subseteq \bigoplus_{i=1}^n E_i$. Thus $\text{supp}(\text{im}(f))$ is finite and hence \mathcal{L} satisfies condition (F) for a family $\{E_i\}_{i \in \mathbb{N}}$. Thus by Lemma 4.2 we have that $\bigoplus_{i \in \mathbb{N}} E_i$ is an \mathcal{L} -injective module. Thus for any family $\{E_i\}_{i \in \mathbb{N}}$ of \mathcal{L} -injective R -modules in \mathcal{K} , we have $\bigoplus_{i \in \mathbb{N}} E_i$ is \mathcal{L} -injective. Since \mathcal{L} satisfies conditions (α) , (μ) and (δ) , thus by Corollary 4.6, we have that every direct sum of \mathcal{L} -injective modules in \mathcal{K} is \mathcal{L} -injective. \square

A nonempty class \mathcal{K} of modules is said to be a natural class if it is closed under submodules, arbitrary direct sums and injective hulls [10]. Examples of natural classes include $R\text{-Mod}$, any hereditary torsionfree classes and stable hereditary torsion classes.

We can now state the following result, found in [21, p. 643] as a corollary.

Corollary 4.13. *Let \mathcal{K} be a natural class of modules closed under isomorphic copies. Then the following statements are equivalent:*

- (1) *every direct sum of injective modules in \mathcal{K} is injective;*
- (2) *$H_{\mathcal{K}}(R)$ has ACC.*

Proof. (1) \Rightarrow (2) By taking $\mathcal{L} = \mathcal{M}$ and applying Lemma 2.11, Lemma 4.9 and Theorem 4.10.

(2) \Rightarrow (1) By taking $\mathcal{L} = \mathcal{M}$ and applying [21, Lemma 7, p. 637] and Theorem 4.12. \square

Corollary 4.14. *Let ρ be any P -filter and let \mathcal{K} be any nonempty class of modules closed under isomorphic copies and submodules. If every ascending chain $J_1 \subseteq J_2 \subseteq \dots$ of left ideals of R such that $(J_{i+1}/J_i) \in \mathcal{K}$, $\forall i \in \mathbb{N}$ and $J_\infty = \bigcup_{i=1}^\infty J_i$ is $s\text{-}\mathcal{L}_{(\rho, \infty)}$ -dense in R terminates, then every direct sum of $\mathcal{L}_{(\rho, \infty)}$ -injective modules in \mathcal{K} is $\mathcal{L}_{(\rho, \infty)}$ -injective.*

Proof. By Lemma 2.11, Lemma 4.9 and Theorem 4.12. \square

Let τ be a hereditary torsion theory. A nonempty class \mathcal{K} of modules is said to be τ -natural class if \mathcal{K} closed under submodules, isomorphic copies, arbitrary direct sums and τ -injective hulls [9, p. 163].

Corollary 4.15. ([9, Proposition 5.3.5, p. 165]) *Let \mathcal{K} be a τ -natural and suppose that every ascending chain $J_1 \subseteq J_2 \subseteq \dots$ of left ideals of R such that $(J_{i+1}/J_i) \in \mathcal{K}$, $\forall i \in \mathbb{N}$ and $J_\infty = \bigcup_{i=1}^\infty J_i$ is τ -dense in R terminates. Then every direct sum of τ -injective modules in \mathcal{K} is τ -injective.*

Proof. Take $\rho = \rho_\tau$ and apply Corollary 4.14. \square

The following corollary, in which we give conditions under which the class of \mathcal{L} -injective modules is closed under direct sums, is one of the main aims of this section.

Corollary 4.16. *Consider the following three conditions:*

- (1) *the class of \mathcal{L} -injective R -modules is closed under direct sums;*
- (2) *every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R with $I_\infty = \bigcup_{j=1}^\infty I_j$ $s\text{-}\mathcal{L}$ -dense in R , terminates;*
- (3) *the following conditions hold:*
 - (a) *every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of $s\text{-}\mathcal{L}$ -dense left ideals of R , terminates;*
 - (b) *for every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R with $I_\infty = \bigcup_{j=1}^\infty I_j$ $s\text{-}\mathcal{L}$ -dense in R , there exists a positive integer n such that I_n is $s\text{-}\mathcal{L}$ -dense in R .*

If \mathcal{L} satisfies conditions (α) and (δ) , then (1) implies (2). Also, (2) implies (3b) and if \mathcal{L} satisfies conditions (α) and (β) , then (2) implies (3a). Moreover, if \mathcal{L} satisfies conditions (α) , (μ) , (δ) and (I), then all above three conditions are equivalent.

Proof. (1) \Rightarrow (2) Let \mathcal{L} satisfy conditions (α) and (δ) . Take $\mathcal{K} = R\text{-Mod}$ and apply Theorem 4.10.

(2) \Rightarrow (3b) Take $\mathcal{K} = R\text{-Mod}$ and apply Proposition 4.11.

(2) \Rightarrow (3a) Let \mathcal{L} satisfy conditions (α) and (β) . Take $\mathcal{K} = R\text{-Mod}$ and apply Proposition 4.11.

(3) \Rightarrow (1) Let \mathcal{L} satisfy conditions (α) , (μ) , (δ) and (I). Take $\mathcal{K} = R\text{-Mod}$. By Proposition 4.11, we have that every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R with $I_\infty = \bigcup_{j=1}^\infty I_j$ $s\text{-}\mathcal{L}$ -dense in R , terminates. Thus every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R such that $(I_{j+1}/I_j) \in \mathcal{K}$, $\forall j \in \mathbb{N}$ and $I_\infty = \bigcup_{j=1}^\infty I_j$ $s\text{-}\mathcal{L}$ -dense in R , terminates. Since \mathcal{K} is closed under

isomorphic copies and submodules, by Theorem 4.12 we have that every direct sum of \mathcal{L} -injective modules in \mathcal{K} is \mathcal{L} -injective. Thus the class of \mathcal{L} -injective R -modules is closed under direct sums. \square

Corollary 4.17. *Let ρ be any P -filter. Then the following statements are equivalent.*

- (1) *The class of $\mathcal{L}_{(\rho, \rho_\infty)}$ -injective R -modules is closed under direct sums.*
- (2) *Every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R with $I_\infty = \bigcup_{j=1}^\infty I_j$ $s\text{-}\mathcal{L}_{(\rho, \rho_\infty)}$ -dense in R , terminates.*
- (3) *The following conditions hold.*
 - (a) *Every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of $s\text{-}\mathcal{L}_{(\rho, \rho_\infty)}$ -dense left ideals of R , terminates.*
 - (b) *For every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of left ideals of R with $I_\infty = \bigcup_{j=1}^\infty I_j$ $s\text{-}\mathcal{L}_{(\rho, \rho_\infty)}$ -dense in R , there exists a positive integer n such that I_n is $s\text{-}\mathcal{L}_{(\rho, \rho_\infty)}$ -dense in R .*
- (4) *For any family $\{E_i\}_{i \in \mathbb{N}}$ of $\mathcal{L}_{(\rho, \rho_\infty)}$ -injective R -modules, $\bigoplus_{i \in \mathbb{N}} E_i$ is $\mathcal{L}_{(\rho, \rho_\infty)}$ -injective.*

Proof. By Lemma 2.11 and Lemma 4.9 we have that $\mathcal{L}_{(\rho, \rho_\infty)}$ satisfies conditions (α) , (μ) , (δ) and (I) . Thus by Corollary 4.16 and Corollary 4.6 we have the equivalence of above four statements. \square

Corollary 4.18. *([9, Theorem 2.3.8, p. 73]) The following statements are equivalent:*

- (1) *R has ACC on τ -dense left ideals and τ is Noetherian;*
- (2) *the class of τ -injective R -modules is closed under direct sums;*
- (3) *the class of τ -injective R -modules is closed under countable direct sums.*

Proof. Take $\rho = \rho_\tau$ and apply Corollary 4.17. \square

5 $\Sigma\text{-}\mathcal{L}$ -Injective Modules

Carl Faith in [13] introduced the concepts of Σ -injectivity and countably Σ -injectivity as follows. An injective module E is said to be Σ -injective if $E^{(A)}$ is injective for any index set A ; E is said to be countably Σ -injective in case $E^{(C)}$ is injective for any countable index set C . Faith in [13] proved that an injective R -module E is Σ -injective if and only if R satisfies ACC on the E -annihilator left ideals if and only if E is countably Σ -injective. S. Charalambides in [7] introduced the concept of $\Sigma\text{-}\tau$ -injectivity and generalized Faith's result.

In this section we introduce the concept of $\Sigma\text{-}\mathcal{L}$ -injectivity as a general case of Σ -injectivity and $\Sigma\text{-}\tau$ -injectivity and prove the result (Theorem 5.4) in which we generalize Faith's result [13, Proposition 3, p. 184] and [7, Theorem 16.16, p. 98].

We start this section with the following definition of a $\Sigma\text{-}\mathcal{L}$ -injective module.

Definition 5.1. *Let E be an \mathcal{L} -injective module. We say that E is $\Sigma\text{-}\mathcal{L}$ -injective if $E^{(A)}$ is \mathcal{L} -injective for any index set A . On other hand, if $E^{(C)}$ is \mathcal{L} -injective for any countable index set C , we say that E is countably $\Sigma\text{-}\mathcal{L}$ -injective.*

The following corollary is a special case of Corollary 4.6, by taking $\mathcal{K} = \{E\}$.

Corollary 5.2. *Consider the following conditions.*

- (1) *E is $\Sigma\text{-}\mathcal{L}$ -injective.*
- (2) *E is countably $\Sigma\text{-}\mathcal{L}$ -injective.*
- (3) *$E^{(\mathbb{N})}$ is \mathcal{L} -injective.*

Then: (1) implies (2) and (2) implies (3). If \mathcal{L} satisfies conditions (α) , (μ) and (δ) , then (2) implies (1). Moreover, if \mathcal{L} satisfies condition (γ) , then (3) implies (2).

The following corollary is immediately from Lemma 2.11 and Corollary 5.2.

Corollary 5.3. *Let ρ_1 and ρ_2 be any two P -filters. Then the following conditions are equivalent for a module E .*

- (1) *E is $\Sigma\text{-}\mathcal{L}_{(\rho_1, \rho_2)}$ -injective.*
- (2) *E is countably $\Sigma\text{-}\mathcal{L}_{(\rho_1, \rho_2)}$ -injective.*
- (3) *$E^{(\mathbb{N})}$ is $\mathcal{L}_{(\rho_1, \rho_2)}$ -injective.*

Let E be a module. A left ideal I of R is said to be an E -annihilator if there is $N \subseteq E$ such that $I = (0 : N) = \{r \in R \mid rN = 0\}$ (i.e., I is the annihilator of a subset of E).

The following theorem is the main result in this section in which we generalize [7, Theorem 16.16, p. 98] and [13, Proposition 3, p. 184].

Theorem 5.4. *Consider the following three conditions for an \mathcal{L} -injective module E :*

- (1) E is countably Σ - \mathcal{L} -injective;
- (2) every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of E -annihilators in R with $I_\infty = \bigcup_{j=1}^\infty I_j$ s - \mathcal{L} -dense in R , terminates;
- (3) The following conditions hold.
 - (a) Every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of E -annihilators in R with I_j being s - \mathcal{L} -dense in R , $\forall j \in \mathbb{N}$, terminates.
 - (b) For every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of E -annihilators in R with $I_\infty = \bigcup_{j=1}^\infty I_j$ s - \mathcal{L} -dense in R , there exists a positive integer n such that I_n is s - \mathcal{L} -dense in R .

Then: if \mathcal{L} satisfies condition (δ) , then (1) implies (2). Also, (2) implies (3b) and if \mathcal{L} satisfies conditions (α) and (β) , then (2) implies (3a). Moreover, if \mathcal{L} satisfies conditions (α) , (μ) , (β) and (I), then (3) implies (1).

Proof. (1) \Rightarrow (2) Let \mathcal{L} satisfy condition (δ) . For the sake of getting a contradiction assume that (2) does not hold. Then there exist E -annihilators I_1, I_2, \dots in R such that $I_1 \subsetneq I_2 \subsetneq \dots$ and $I_\infty = \bigcup_{j=1}^\infty I_j$ is s - \mathcal{L} -dense in R . Hence we have the following descending chain $r_E(I_1) \supsetneq r_E(I_2) \supsetneq \dots$. For every $n \in \mathbb{N}$, choose $x_n \in r_E(I_n) - r_E(I_{n+1})$. Thus $x = (x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$. Define $f : I_\infty \rightarrow E^{\mathbb{N}}$ by $f(a) = ax, \forall a \in I_\infty$. It is clear that f is a homomorphism. For a fixed $a \in I_\infty$ let n be the smallest positive integer such that $a \in I_n$. Then, for every $k \geq 0$, $a \in I_n \subseteq I_{n+k}$. Since $x_{n+k} \in r_E(I_{n+k})$, thus $ax_{n+k} = 0, \forall k \geq 0$. Hence $ax \in E^{(\mathbb{N})}$. Thus f is a homomorphism from I_∞ into $E^{(\mathbb{N})}$. Since I_∞ is s - \mathcal{L} -dense in R , thus $(R, I_\infty, I_\infty, I_\infty) \in \mathcal{L}$. Since \mathcal{L} satisfies condition (δ) , thus $(R, I_\infty, f, E^{(\mathbb{N})}) \in \mathcal{L}$. Since $E^{(\mathbb{N})}$ is \mathcal{L} -injective, thus by Theorem 2.12, there exists an element $y \in E^{(\mathbb{N})}$ such that $f(a) = ay, \forall a \in I_\infty$. Since $y \in E^{(\mathbb{N})}$, thus $y = (y_1, y_2, \dots, y_t, 0, 0, \dots)$, for some $t \in \mathbb{N}$. Since $ax = f(a) = ay, \forall a \in I_\infty$, thus $(ax_1, ax_2, \dots) = (ay_1, ay_2, \dots, ay_t, 0, 0, \dots)$ and this implies that $ax_{t+1} = 0, \forall a \in I_\infty$. Thus $x_{t+1} \in r_E(I_\infty)$. Since $I_{t+2} \subsetneq I_\infty$, thus $r_E(I_\infty) \subseteq r_E(I_{t+2})$ and so $x_{t+1} \in r_E(I_{t+2})$. This contradicts the fact that $x_{t+1} \in r_E(I_{t+1}) - r_E(I_{t+2})$.

(2) \Rightarrow (3b) Let $I_1 \subseteq I_2 \subseteq \dots$ be any ascending chain of E -annihilators in R with $I_\infty = \bigcup_{j=1}^\infty I_j$ s - \mathcal{L} -dense in R . By hypothesis, there exists a positive integer n such that $I_n = I_{n+k}, \forall k \in \mathbb{N}$ and so $I_n = I_\infty$. Hence I_n is s - \mathcal{L} -dense in R .

(2) \Rightarrow (3a) Let \mathcal{L} satisfy conditions (α) and (β) and let $I_1 \subseteq I_2 \subseteq \dots$ be any ascending chain of E -annihilators in R , such that the I_j are s - \mathcal{L} -dense left ideals of R . Since $I_1 \subseteq I_\infty$ and \mathcal{L} satisfies conditions (α) and (β) , thus by Lemma 4.8 we have that I_∞ is a s - \mathcal{L} -dense left ideal of R . By hypothesis, the chain $I_1 \subseteq I_2 \subseteq \dots$ terminates.

(3) \Rightarrow (1) Let \mathcal{L} satisfy conditions (α) , (μ) , (β) and (I) and let $(R, J, f, E^{(\mathbb{N})}) \in \mathcal{L}$. Since E is \mathcal{L} -injective, by Proposition 2.6 we have that $E^{\mathbb{N}}$ is \mathcal{L} -injective. Since $E^{(\mathbb{N})}$ is a submodule of $E^{\mathbb{N}}$, thus $g = i \circ f : J \rightarrow E^{\mathbb{N}}$ is a homomorphism, where $i : E^{(\mathbb{N})} \rightarrow E^{\mathbb{N}}$ is the inclusion homomorphism. Since \mathcal{L} satisfies condition (β) , $(R, J, i \circ f, E^{(\mathbb{N})}) \in \mathcal{L}$. Thus by Theorem 2.12, there is an element $x = (x_1, x_2, \dots) \in E^{\mathbb{N}}$ such that $g(a) = ax, \forall a \in J$. Thus $f(a) = i(f(a)) = (i \circ f)(a) = g(a) = ax, \forall a \in J$. Let $X = \{x_1, x_2, \dots\}$ and $X_k = X \setminus \{x_1, x_2, \dots, x_k\} = \{x_{k+1}, x_{k+2}, \dots\}$, for all $k \geq 1$. Thus we have the following descending chain of subsets of X : $X \supseteq X_1 \supseteq X_2 \supseteq \dots$; this yields an ascending chain of E -annihilators in R : $l_R(X) \subseteq l_R(X_1) \subseteq l_R(X_2) \subseteq \dots$. Let $J_{k+1} = l_R(X_k)$, for all $k \geq 0$, where $X_0 = X$ and $J_\infty = \bigcup_{i=1}^\infty J_i$. Since $f(J) \subseteq E^{(\mathbb{N})}$, for any $a \in J$, either $ax_k = 0, \forall k \in \mathbb{N}$, or there is a largest integer $n \in \mathbb{N}$ such that $ax_n \neq 0$. If there is a largest integer $n \in \mathbb{N}$ such that $ax_n \neq 0$, then $ax_{n+k} = 0, \forall k \geq 1$. Therefore $a \in l_R(X_n) = J_{n+1} \subseteq J_\infty$. Thus for any $a \in J$, we have $a \in J_\infty$, and this implies that $J \subseteq J_\infty$. Since $(R, J, f, E^{(\mathbb{N})}) \in \mathcal{L}$ and \mathcal{L} satisfies condition (I), thus J is s - \mathcal{L} -dense left ideal in R . Since $J \subseteq J_\infty$ and \mathcal{L} satisfies conditions (α) and (β) , thus by Lemma 4.8 we have that J_∞ is s - \mathcal{L} -dense left ideal in R . Thus we have the following ascending chain $J_1 \subseteq J_2 \subseteq \dots$ of E -annihilators in R such that J_∞ is s - \mathcal{L} -dense left ideal in R . By applying condition (3b), there is $s \in \mathbb{N}$ such that J_s is s - \mathcal{L} -dense left ideal in R . Since $J_s \subseteq J_{s+k}, \forall k \in \mathbb{N}$ and \mathcal{L} satisfies conditions (α) and (β) , thus by Lemma 4.8 we have that J_{s+k} is s - \mathcal{L} -dense left ideal in $R, \forall k \in \mathbb{N}$. Thus we have the following ascending chain $J_s \subseteq J_{s+1} \subseteq \dots$ of E -annihilators in R such that J_{s+k} is s - \mathcal{L} -dense left ideal in $R, \forall k \in \mathbb{N}$. By applying condition (3a), the chain $J_s \subseteq J_{s+1} \subseteq \dots$ becomes stationary at a left ideal of R , say $J_t = l_R(X_{t-1})$ and so $J_t = J_\infty$. Thus for any $a \in J$, we have $ax_{t+k} = 0, \forall k \geq 0$ and then $a(0, 0, \dots, 0, x_t, x_{t+1}, \dots) = 0$. Take $y = (x_1, x_2, \dots, x_{t-1}, 0, 0, \dots)$. It is clear that $y \in E^{(\mathbb{N})}$ and for any $a \in J$, then $f(a) = ax = ax - a(0, 0, \dots, 0, x_t, x_{t+1}, 0, 0, \dots) = a(x_1, x_2, \dots, x_{t-1}, 0, 0, \dots) = ay$. Thus

for every $(R, J, f, E^{(\mathbb{N})}) \in \mathcal{L}$, there exists an element $y \in E^{(\mathbb{N})}$ such that $f(a) = ay, \forall a \in J$. Since \mathcal{L} satisfies conditions (α) and (μ) , thus $E^{(\mathbb{N})}$ is \mathcal{L} -injective, by Theorem 2.12. Since \mathcal{L} satisfies condition (γ) , thus E is countably Σ - \mathcal{L} -injective, by Corollary 5.2. \square

Corollary 5.5. *Let ρ be any P -filter. Then the following conditions are equivalent.*

- (1) *E is countably Σ - $\mathcal{L}_{(\rho, \infty)}$ -injective.*
- (2) *Every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of E -annihilators in R with $I_\infty = \bigcup_{j=1}^\infty I_j$ is s - $\mathcal{L}_{(\rho, \infty)}$ -dense left ideal in R , terminates.*
- (3) *The following conditions hold.*
 - (a) *Every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of E -annihilators in R with I_j is s - $\mathcal{L}_{(\rho, \infty)}$ -dense left ideals of $R, \forall j \in \mathbb{N}$, terminates.*
 - (b) *For every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of E -annihilators in R with $I_\infty = \bigcup_{j=1}^\infty I_j$ is s - $\mathcal{L}_{(\rho, \infty)}$ -dense left ideal in R , there exists a positive integer n such that I_n is s - $\mathcal{L}_{(\rho, \infty)}$ -dense in R .*
- (4) *E is Σ - $\mathcal{L}_{(\rho, \infty)}$ -injective.*

Proof. By Lemma 2.11, Lemma 4.9 and Theorem 5.4, we have the equivalence of (1), (2) and (3). \square

(1) \Leftrightarrow (4) By Corollary 5.2. \square

Corollary 5.6. *([7, Theorem 16.16, p. 98]) Let τ be any hereditary torsion theory and let E be τ -injective module. Then the following conditions are equivalent.*

- (1) *E is countably Σ - τ -injective.*
- (2) *Every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of E -annihilators in R with $I_\infty = \bigcup_{j=1}^\infty I_j$ is τ -dense left ideal in R , terminates.*
- (3) *The following conditions hold.*
 - (a) *Every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of E -annihilators in R with I_j is τ -dense left ideals of $R, \forall j \in \mathbb{N}$, terminates.*
 - (b) *For every ascending chain $I_1 \subseteq I_2 \subseteq \dots$ of E -annihilators in R with $I_\infty = \bigcup_{j=1}^\infty I_j$ is τ -dense left ideal in R , there exists a positive integer n such that I_n is τ -dense in R .*
- (4) *E is Σ - τ -injective.*

Proof. By taking a P -filter $\rho = \rho_\tau$ and applying Corollary 5.5. \square

Corollary 5.7. *([13, Proposition 3, p. 184]) The following conditions on an injective module E are equivalent.*

- (1) *E is countably Σ -injective.*
- (2) *R satisfies the ACC on the E -annihilators left ideals.*
- (3) *E is Σ -injective.*

Proof. By taking $\rho = \mathfrak{R}$ and applying Corollary 5.5. \square

Corollary 5.8. *Let \mathcal{L} satisfy conditions (α) , (μ) and (δ) and let $\{E_i \mid 1 \leq i \leq n\}$ be a family of modules. If E_i is Σ - \mathcal{L} -injective, $\forall i = 1, 2, \dots, n$, then $\bigoplus_{i=1}^n E_i$ is Σ - \mathcal{L} -injective.*

Proof. Since E_i is Σ - \mathcal{L} -injective, $\forall i = 1, 2, \dots, n$, thus $E_i^{(\mathbb{N})}$ is \mathcal{L} -injective, $\forall i = 1, 2, \dots, n$. Thus by Corollary 2.7, we have that $\bigoplus_{i=1}^n E_i^{(\mathbb{N})}$ is \mathcal{L} -injective. Since $(\bigoplus_{i=1}^n E_i)^{(\mathbb{N})} = (E_1 \oplus E_2 \oplus \dots \oplus E_n)^{(\mathbb{N})} = E_1^{(\mathbb{N})} \oplus E_2^{(\mathbb{N})} \oplus \dots \oplus E_n^{(\mathbb{N})} = \bigoplus_{i=1}^n E_i^{(\mathbb{N})}$, thus $(\bigoplus_{i=1}^n E_i)^{(\mathbb{N})}$ is \mathcal{L} -injective. Hence $\bigoplus_{i=1}^n E_i$ is Σ - \mathcal{L} -injective, by Corollary 5.2. \square

Corollary 5.9. *Let ρ_1 and ρ_2 be any two P -filters and let $\{E_i \mid 1 \leq i \leq n\}$ be a family of modules. If E_i is Σ - $\mathcal{L}_{(\rho_1, \rho_2)}$ -injective, $\forall i = 1, 2, \dots, n$, then $\bigoplus_{i=1}^n E_i$ is Σ - $\mathcal{L}_{(\rho_1, \rho_2)}$ -injective.*

Proof. By Lemma 2.11 and Corollary 5.8. \square

We can now state the following result, found in [7, p. 98] as a corollary.

Corollary 5.10. *Let τ be any hereditary torsion theory and let $\{E_i \mid 1 \leq i \leq n\}$ be a family of modules. If E_i is Σ - τ -injective, $\forall i = 1, 2, \dots, n$, then $\bigoplus_{i=1}^n E_i$ is Σ - τ -injective.*

Proof. By taking the two P -filters $\rho_1 = \rho_\tau$ and $\rho_2 = \mathfrak{R}$ and applying Corollary 5.9. \square

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